A generalized theorem on the average array directivity factor

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Abstract

The beampattern of an array consisting of N elements is determined by the beampatterns of the individual elements, their placement, and the weights assigned to them. For each look direction, it is possible to design weights that maximize the array directivity factor (DF). For the case of an array of omnidirectional elements using optimal weights, it has been shown that the average DF over all look directions equals the number of elements. The validity of this theorem is not dependent on array geometry. We generalize this theorem by means of an alternative proof. The chief contributions of this letter are: (a) a compact and direct proof, (b) generalization to arrays containing directional elements (such as cardioids and dipoles), and (c) generalization to arbitrary wave propagation models. A discussion of the theorem's ramifications on array processing is provided.

I. Introduction

In array processing, signals corresponding to different elements are multiplied by a set of weights and combined to aid the transmission or reception of signals. The signals corresponding to the different elements may interfere constructively or destructively when combined, resulting in different net signal levels for different directions or locations. These variations in signal level are described by the array *beampattern* and the processing technique is known as *beamforming*. Applications of beamforming are widespread [1] and encompass such areas as radar, sonar, communications, seismology, astronomy, oceanography, medical tomography, and acoustic signal processing.

The properties of the beampattern are affected by the positions of the array elements, the beampatterns corresponding to each individual element (e.g., omnidirectional or cardioid), and the weights assigned to each element. While the positions and beampatterns of each element are determined by the array construction and cannot easily be changed, the weights can often be easily modified online. This degree of flexibility can be used to steer the beampattern towards different look directions.

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It is often desirable to design a beamformer which transmits or receives a signal from one particular direction while minimizing the power corresponding to all other directions. The *directivity factor (DF)* is used as a performance measure which is defined as the power corresponding to the desired direction divided by the ambient power (averaged over all directions)[1]. For a given array constellation (with predetermined element positions and beampatterns), it is possible to select weights which maximize the DF. This can be achieved by using minimum variance distortionless response (MVDR) weights designed for diffuse noise [2].

One type of array constellation which is quite common in the literature consists of N omnidirectional elements separated by distances which are integer multiples of half a wavelength (such as a uniform linear array or a tetrahedral array). For this case, the MVDR weights which produce the maximum DF correspond to a delay and sum beamformer [3], [1]. The array's DF is N for all look directions at this particular wavelength. This property is *atypical* and is valid specifically for the described constellations.

In general, the optimal DF afforded by MVDR beamforming does not equal the number of array elements [4] and can be highly dependent on the desired direction-of-arrival (DOA). For instance, consider a linear array consisting of N omnidirectional elements with vanishingly small distances between elements. The maximal DF attainable in theory by this array at an *arbitrary* look direction can be expressed as a sequence containing Legendre polynomials which is dependent on the DOA [3]. At *endfire*, the DF indicated by this sequence equals the superdirective value of N^2 [5] which is the highest DF known to be attainable for any configuration of omnidirectional elements [6]. For the *broadside* look direction, the sequence can be approximated as $4(\lceil N/2 \rceil - 1)/\pi$ [7]. This example illustrates the dependence of the DF upon the desired DOA.

It is desirable to produce an array capable of maintaining a high DF for all look directions. Gilbert and Morgan [3] presented a theorem which states that for any array configuration of omnidirectional elements, the optimal DF averaged over all directions is N. This implies that if some look directions can attain a DF which is higher than N, other directions will have a DF which is lower than N; it is impossible to construct an array which maintains a DF simultaneously higher than N for all look directions. Averaging the maximum DF over all look directions yields an immutable value which cannot be modified by altering the array configuration.

As the theorem was stated for an array of omnidirectional elements, it is natural to inquire whether a higher value of average DF can be obtained for an array incorporating directional elements. For example, can an array which contains cardioid and dipole elements achieve an average DF greater than N? In this letter, we generalize Gilbert and Morgan's theorem and show that it is valid for an array for which the beampatterns of the individual elements are arbitrary and not necessarily identical. We do this by means of an alternative and compact proof. The chief contributions of this letter are: (a) a concise and direct proof, (b) generalization to arrays containing directional elements (such as cardioids and dipoles), and (c) generalization to arbitrary wave propagation models.

This letter is structured as follows. Section II defines terms and notation used in the correspondence, and provides background pertaining to maximizing the DF. In Sec. III we present the theorem and its proof. Section IV provides

[1]This quantity is also refereed to simply as *directivity*, in particular in the antenna community.

examples of cases where the theorem may be applied and where it may not, and discusses generalized wave propagation models and ramifications of the theorem for the field of beamforming. Section V concludes with a brief summary.

II. Notation and background

The theory of beamforming applies to both transmitting and receiving arrays. In the sequel, our discussion proceeds from the viewpoint of a receiving array without loss of generality.

A. Notation and definitions

Consider an array consisting of N sensors receiving a desired signal in a noisy environment. In the frequency domain, the N sensor signals are denoted $x_{1\dots N}(\omega)$, where ω denotes angular frequency. The response of the n-th sensor to a plane wave (i.e., the elements's beampattern) is denoted $b_n(\mathbf{u},\omega)$, where $\mathbf{u}$ is a unit-vector corresponding to the DOA[2]. These element beampatterns are concatenated into a column-vector $\mathbf{b}(\mathbf{u},\omega) = [b_1(\mathbf{u},\omega)\dots b_N(\mathbf{u},\omega)]^T$. The sensor positions are denoted by the matrix $\mathbf{P}$ with dimensions $3\times N$. The steering-vector portrays the overall array response which consists of the individual sensors' beampatterns as well as phase shifts due to propagation delays. It is given by:

$$\mathbf{v}(\mathbf{u},\omega) = \mathbf{b}(\mathbf{u},\omega)\odot\exp\{j\,k\cdot\mathbf{P}^T\mathbf{u}\}\,, \tag{1}$$

where the operator $\odot$ represents the Hadamard (element-wise) product and $k = \omega/c$ is the wavenumber (with c corresponding to the velocity of wave propagation). When a signal $s(\omega)$ arrives from the direction $\mathbf{u}_0$, the sensors' signals are given by:

$$\mathbf{x}(\omega) = \mathbf{v}(\mathbf{u}_0,\omega)s(\omega) + \mathbf{n}(\omega)\,, \tag{2}$$

where $\mathbf{x}(\omega) = [x_1(\omega)\dots x_N(\omega)]^T$, $s(\omega)$ is the desired signal, and $\mathbf{n}(\omega)$ is noise.

The beamformer produces an output signal by performing a weighted sum of the input channels:

$$y(\omega) = \mathbf{w}^H(\omega)\mathbf{x}(\omega)\,, \tag{3}$$

where $\mathbf{w}(\omega) = [w_1(\omega)\dots w_N(\omega)]^T$ contains the weights corresponding to each sensor. Substituting (2) into (3) shows that the desired signal is scaled by a factor of $\mathbf{w}^H(\omega)\mathbf{v}(\mathbf{u}_0,\omega)$. The beampattern[3]:

$$\mathrm{BP}(\mathbf{u}) = \mathbf{w}^H\mathbf{v}(\mathbf{u})\,, \tag{4}$$

describes changes in the signal level as a function of its DOA. Similarly, the beam-power $|\mathrm{BP}(\mathbf{u})|^2$ describes changes in a signal's power as a function of DOA.

The level of noise at the output signal is given by:

$$E\{|\mathbf{w}^H\mathbf{n}|^2\} = \mathbf{w}^H E\{\mathbf{n}\mathbf{n}^H\}\mathbf{w} = \mathbf{w}^H\mathbf{R}\mathbf{w}\,, \tag{5}$$

[2]It should be noted that $b_n(\mathbf{u},\omega)$ is a scalar quantity. For electromagnetic fields, this holds when all array elements share the same polarization.

[3]In the interest of conciseness, explicit dependence on frequency is dropped from this point onwards.

where $E\{\cdot\}$ represents statistical expectation and $\mathbf{R} = E\{\mathbf{n}\mathbf{n}^H\}$ is the noise covariance matrix. The beamformer which minimizes the noise level while maintaining a unity response in the look direction $\mathbf{u}_0$ uses the MVDR weights, which are given by [8]:

$$\mathbf{w}_{\mathrm{MVDR}}(\mathbf{u}_0) = \frac{\mathbf{R}^{-1}\mathbf{v}_0}{\mathbf{v}_0^H\mathbf{R}^{-1}\mathbf{v}_0}\,, \tag{6}$$

where $\mathbf{v}_0 = \mathbf{v}(\mathbf{u}_0)$ is the steering vector corresponding to the look direction.

B. The directivity factor and its maximization

The DF of a beamformer describes the ratio of beam-power in the desired DOA relative to the average beam-power over all directions. A high DF corresponds to sharper and more focused beams. For a beamformer with a set of weights $\mathbf{w}$, the spherical DF is defined as:

$$\Gamma_{\text{sph}}(\mathbf{w},\mathbf{u}_0) = \frac{|\mathbf{w}^H\mathbf{v}(\mathbf{u}_0)|^2}{\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}|\mathbf{w}^H\mathbf{v}(\mathbf{u})|^2\,\sin(\theta)d\theta\,d\phi}\,, \tag{7}$$

where the DOA $\mathbf{u} = [\cos(\phi)\sin(\theta),\, \sin(\phi)\sin(\theta),\, \cos(\theta)]^T$ is specified by azimuth and inclination parameters. The spherical DF of (7) is the standard definition for DF. On occasion, only DOAs within the *xy*-plane (i.e. $\theta = \frac{\pi}{2}$) are of interest. In this case, an alternative version – cylindrical DF – is defined:

$$\Gamma_{\text{cyl}}(\mathbf{w},\mathbf{u}_0) = \frac{|\mathbf{w}^H\mathbf{v}(\mathbf{u}_0)|^2}{\frac{1}{2\pi}\int_0^{2\pi}|\mathbf{w}^H\mathbf{v}(\mathbf{u})|^2\,d\phi}\,. \tag{8}$$

Both (7) and (8) subscribe to the general form:

$$\Gamma_{\text{gen}}(\mathbf{w},\mathbf{q}_0) = \frac{|\mathbf{w}^H\mathbf{v}(\mathbf{q}_0)|^2}{\kappa\int_{\mathbf{q}\in\mathcal{Q}}|\mathbf{w}^H\mathbf{v}(\mathbf{q})|^2\,A(\mathbf{q})d\mathbf{q}}\,, \tag{9}$$

where the steering-vectors are specified by the parameter vector $\mathbf{q}$, integration is performed over some set of interest $\mathcal{Q}$ in the parameter space, and $A(\mathbf{q})$ is a function denoting the weight for each instance of $\mathbf{q}$ [such as the Jacobian $\sin(\theta)$ of (7), or some measure of likelihood]. The normalizing constant κ is defined as $1/\int_{\mathbf{q}\in\mathcal{Q}}A(\mathbf{q})d\mathbf{q}$. Equation (9) is used in Sec. IV to formulate generalizations of the DF.

The denominator of (9) can be expressed as:

$$\mathbf{w}^H\left(\kappa\int_{\mathbf{q}\in\mathcal{Q}}\mathbf{v}(\mathbf{q})\mathbf{v}^H(\mathbf{q})\,A(\mathbf{q})d\mathbf{q}\right)\mathbf{w} = \mathbf{w}^H\boldsymbol{\Phi}\mathbf{w}\,, \tag{10}$$

where $\boldsymbol{\Phi}$ is defined as:

$$\boldsymbol{\Phi} = \kappa\int_{\mathbf{q}\in\mathcal{Q}}\mathbf{v}(\mathbf{q})\mathbf{v}^H(\mathbf{q})\,A(\mathbf{q})d\mathbf{q}\,. \tag{11}$$

From the fact that the denominator of (9) is nonnegative, it is evident that $\boldsymbol{\Phi}$ must be positive semidefinite. It may be assumed that $\boldsymbol{\Phi}$ is nonsingular[4] and is hence positive definite. Substituting (10) into (9) yields:

$$\Gamma_{\text{gen}}(\mathbf{w},\mathbf{q}_0) = \frac{|\mathbf{w}^H\mathbf{v}(\mathbf{q}_0)|^2}{\mathbf{w}^H\boldsymbol{\Phi}\mathbf{w}}\,. \tag{12}$$

[4]A singular $\boldsymbol{\Phi}$ indicates that the array contains at least one sensor which is redundant in the sense that it does not provide further directional information. Removal of redundant sensors will produce a smaller nonsingular matrix whose rank is identical to that of the original $\boldsymbol{\Phi}$ (see Sec. IV-A for an example).

In order to attain maximal DF, one must select the optimal weights $\mathbf{w}_{\text{opt}}(\mathbf{q}_0)$ which maximize (12). The well-known solution [3], [8], [9] to this optimization problem is $\mathbf{w}_{\text{opt}}(\mathbf{q}_0) = \lambda \mathbf{\Phi}^{-1}\mathbf{v}(\mathbf{q}_0)$, where λ is any nonzero constant. An MVDR beamformer (6) which uses $\mathbf{R} = \mathbf{\Phi}$ corresponds to $\mathbf{w}_{\text{opt}}(\mathbf{q}_0)$ [with $\lambda = (\mathbf{v}_0^H \mathbf{\Phi}^{-1}\mathbf{v}_0)^{-1}$] and reduces the impact of a diffuse noise field[5]. Substituting $\mathbf{w}_{\text{opt}}(\mathbf{q}_0)$ into (12) yields the maximum DF, which is given by:

$$\Gamma_{\text{max}}(\mathbf{q}_0) = \mathbf{v}^H(\mathbf{q}_0)\mathbf{\Phi}^{-1}\mathbf{v}(\mathbf{q}_0)\,. \tag{13}$$

III. Average directivity factor

In this Section, we formally state the theorem for the average DF and then provide a compact proof.

Theorem: *For an array consisting of N elements using weights attaining the maximum DF for each look direction, the average DF over all look directions is N.*

Proof: The average DF can be expressed as:

$$\begin{aligned}\bar{\Gamma}_{\text{max}} &= \kappa \int_{\mathbf{q}\in\mathcal{Q}} \Gamma_{\text{max}}(\mathbf{q})\, A(\mathbf{q}) d\mathbf{q} \\ &= \kappa \int_{\mathbf{q}\in\mathcal{Q}} \mathbf{v}^H(\mathbf{q})\mathbf{\Phi}^{-1}\mathbf{v}(\mathbf{q})A(\mathbf{q})\, d\mathbf{q}\,.\end{aligned} \tag{14}$$

Since $\mathbf{v}^H(\mathbf{q})\mathbf{\Phi}^{-1}\mathbf{v}(\mathbf{q})$ is a scalar, it is equal to its own trace. Hence,

$$\begin{aligned}\bar{\Gamma}_{\text{max}} &= \kappa \int_{\mathbf{q}\in\mathcal{Q}} \text{tr}\left\{\mathbf{v}^H(\mathbf{q})\mathbf{\Phi}^{-1}\mathbf{v}(\mathbf{q})\right\} A(\mathbf{q})\, d\mathbf{q} \\ &= \kappa \int_{\mathbf{q}\in\mathcal{Q}} \text{tr}\left\{\mathbf{\Phi}^{-1}\mathbf{v}(\mathbf{q})\mathbf{v}^H(\mathbf{q})\right\} A(\mathbf{q})\, d\mathbf{q} \\ &= \text{tr}\left\{\mathbf{\Phi}^{-1}\left(\kappa \int_{\mathbf{q}\in\mathcal{Q}} \mathbf{v}(\mathbf{q})\mathbf{v}^H(\mathbf{q})A(\mathbf{q})\, d\mathbf{q}\right)\right\}\,,\end{aligned} \tag{15}$$

where the second line follows from the property that the trace of a product is invariant to a cyclic permutation of the order of multiplicands, and the last line follows from $\text{tr}\{\cdot\}$ being a linear operator. The term in parentheses is $\mathbf{\Phi}$ by definition. Therefore,

$$\bar{\Gamma}_{\text{max}} = \text{tr}\{\mathbf{\Phi}^{-1}\mathbf{\Phi}\} = \text{tr}\{\mathbf{I}\} = N\,, \tag{16}$$

where $\mathbf{I}$ is an $N \times N$ identity matrix with trace N. ∎

The stated proof assumes that $\mathbf{\Phi}^{-1}$ exists (i.e., $\mathbf{\Phi}$ is invertible). In the degenerate case where $\mathbf{\Phi}$ is a singular matrix, the theorem does not hold as we demonstrate in Sec. IV. Our formulation does not assume that the beampatterns of individual elements are omnidirectional, and the proof is valid for an arbitrary selection of element beampatterns.

The average DF theorem follows from the fact that integrals which perform the averagings in (9) and (14) are of the same mold. This leads to $\bar{\Gamma}_{\text{max}} = \text{tr}\{\mathbf{\Phi}^{-1}\mathbf{\Phi}\} = N$. However, for some applications it is possible that the designer is not interested in all steering vectors, but rather wishes to attain high DFs for a certain set of steering vectors $\mathcal{Q}' \neq \mathcal{Q}$ [and in a similar vein, the designer may use a weighting function $A'(\mathbf{q})$ in (14) which differs from that of

[5]The similarity between $\mathbf{w}_{\text{MVDR}}(\mathbf{q}_0)$ and $\mathbf{w}_{\text{opt}}(\mathbf{q}_0)$ is not coincidental. The matrix $\mathbf{\Phi}$ is formed by a linear combination of components possessing the form $\mathbf{v}(\mathbf{q})\mathbf{v}^H(\mathbf{q})$ for all $\mathbf{q} \in \mathcal{Q}$. Hence, it corresponds to waves propagating from all directions constituting a diffuse noise field. The covariance matrix of a diffuse noise field $\mathbf{R}_{\text{dif}}$ is a scaled version of $\mathbf{\Phi}$ (i.e., $\mathbf{R}_{\text{dif}} = \text{const}\,\mathbf{\Phi}$).

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The final version of record is available at http://dx.doi.org/10.1109/LSP.2013.2271722

(9)]. For example, a land-based array receiving signals from sea vessels may require a high DF for the half-plane $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ while reducing diffuse noise originating from all directions. Another example is an array recording a concert performance where the desired signals arrive from the stage and noise comes from the audience [2]. In cases where the two averaging procedures differ, we have:

$$\bar{\Gamma}_{\text{max}} = \text{tr}\{\mathbf{\Phi}^{-1}\mathbf{\Phi}'\}, \tag{17}$$

which in general does not equal N, and may be larger or smaller.

IV. Examples and discussion

In this Section, we provide examples demonstrating where the theorem may be applied and where it may not. We discuss the theorem's relevance for generalized wave propagation models and examine the theorem's implications for the field of array processing.

A. Examples

Let us examine the performance of an acoustic vector-sensor operating in a spherical diffuse noise field. A vector-sensor is an array which consists of $N = 4$ collocated elements: one omnidirectional sensor and three orthogonally oriented dipole sensors. The corresponding steering vector is $\mathbf{v}(\mathbf{u}) = [1 \quad \mathbf{u}^T]^T$, and the diffuse noise matrix is $\mathbf{\Phi} = \text{diag}([1 \quad \frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3}])$ [10]. Application of (13) with these values yields $\Gamma_{\text{opt}}(\mathbf{u}) = 4$. This result has been presented by several authors [11], [12], [13]. Since the optimal DF value is not dependent on the look direction, it follows that $\bar{\Gamma}_{\text{max}} = 4$ in compliance with the theorem of Sec. III.

Next, we examine an acoustic vector-sensor operating in a cylindrical diffuse field. The desired signal is located in the *xy*-plane (i.e., $\theta = \frac{\pi}{2}$), and the look directions are restricted to this region. Therefore, the z coordinate of $\mathbf{u}$ is zero and $\mathbf{v} = [1 \ \cos(\phi) \ \sin(\phi) \ 0]^T$. The diffuse matrix is now $\mathbf{\Phi} = \text{diag}([1 \quad \frac{1}{2} \quad \frac{1}{2} \quad 0])$ [10]. The dipole sensor oriented in the z direction receives neither signal nor noise and thus has no influence on the beamforming process (i.e., its weight w_4 has no impact on the output signal). For all look directions in the *xy*-plane, we obtain $\Gamma_{\text{cyl}}(\mathbf{u}) = 3$. Hence, $\bar{\Gamma}_{\text{max}} = 3 < N$. The average directivity factor theorem is not applicable here since $\mathbf{\Phi}$ is a singular matrix.

In Sec. III, we noted that when a specific set of look directions $\mathcal{Q}'$ is of interest which is not identical to $\mathcal{Q}$, then the average directivity theorem is not valid. To illustrate this point, we examine the case of a circular array of omnidirectional sensors situated in a spherical diffuse noise field. The array consists of $N = 6$ sensors and the circle's diameter is 0.4 wavelengths. The directions of interest are limited to a specific inclination angle θ_0 and encompass all azimuthal angles, i.e.: $\mathcal{Q}' = \{[\phi \quad \theta]^T | \ 0 \le \phi < 2\pi, \theta = \theta_0\}$; at the angle $\theta_0 = 90°$, the DOAs lie in the same plane as the array elements. The average DF using optimal weights is plotted in Fig. 1 as a function of θ_0. The values were obtained by calculating the integral in (14) numerically over all $\mathbf{q} \in \mathcal{Q}'$ (rather than $\mathcal{Q}$). The values vary with θ_0 and can assume values which are significantly below or above $N = 6$.

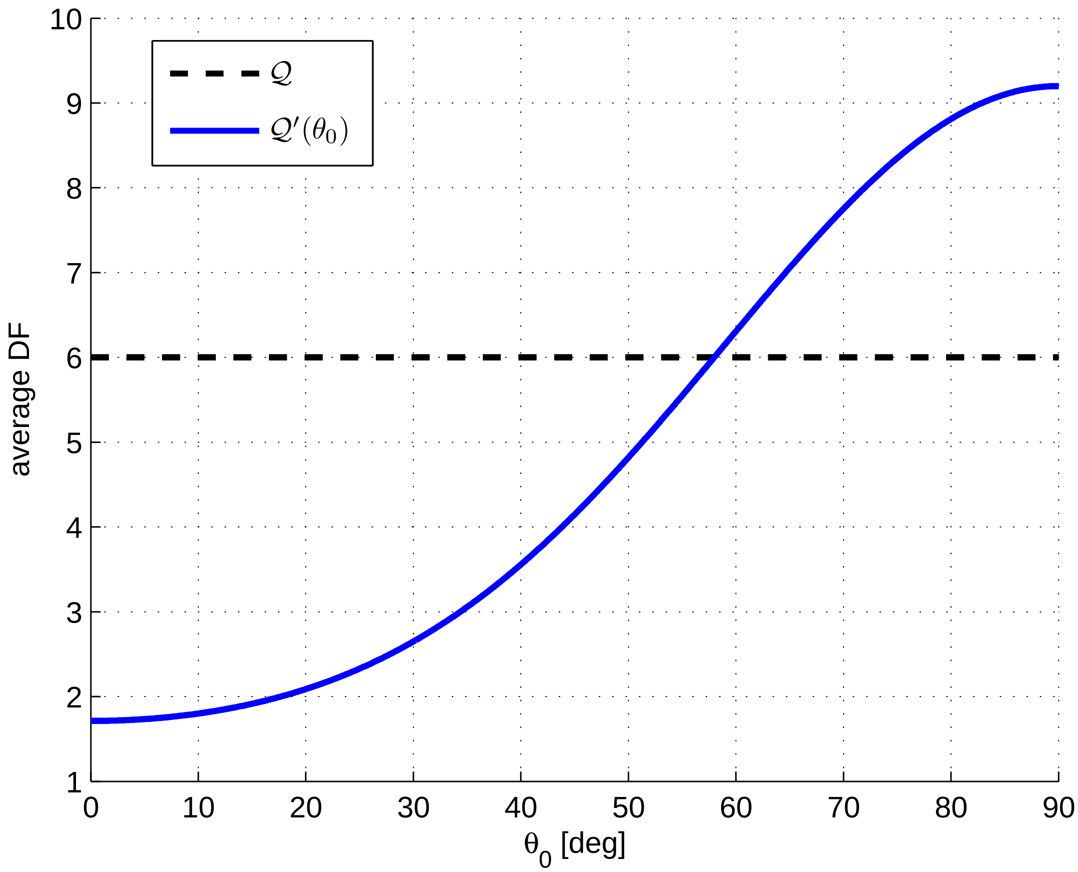

Fig. 1. The average DF using optimal weights is depicted for a circular array consisting of $N = 6$ omnidirectional sensors. The diameter of the array is 0.4 wavelengths and the array operates in a spherical diffuse noise field. The dashed black line represents the average over the entire unit sphere, i.e.: $\mathcal{Q}' = \{[\phi \quad \theta]^T | \, 0 \le \phi < 2\pi, \, 0 \le \theta \le \pi\}$. The solid blue line represents the average over all directions corresponding to inclination θ_0, i.e.: $\mathcal{Q}'(\theta_0) = \{[\phi \quad \theta]^T | \, 0 \le \phi < 2\pi, \, \theta = \theta_0\}$.

B. Discussion

It should be noted that the formulation used to define the DF in (9) is general. The steering-vectors are specified by an arbitrary set of parameters $\mathbf{q}$. In (1), the parameters were DOAs (specified by $\mathbf{u}$) and the steering vectors corresponded to plane waves propagating in a free-field. This conforms with the conventional definition of the DF (7). However, (9) can be generalized to correspond to different wave propagation models. For instance, near-field wave propagation may be used with $\mathbf{q} = [x \; y \; z]^T$ and $\mathcal{Q}$ being some region of interest in Cartesian space. The average DF theorem remains valid for this model. Similarly, $\mathbf{v}(\mathbf{q})$ may incorporate such effects as reverberation and diffraction.

Since the average DF depends on the number of elements and not on their characteristics, it may appear that there is no benefit in using an array containing directional elements. This is misleading since it is often desirable to maintain a high DF only for particular regions; the theorem does not relate to this scenario. Furthermore, the theorem does not make any assumptions on the array robustness. MVDR beamformers often suffer from high sensitivity to perturbations from nominal values. This sensitivity can be reduced by employing directional elements.

V. Conclusion

We extended a theorem stating that the average DF over all look directions for an array using weights which attain the maximal DF at each particular look direction is equal to the number of sensors. A compact proof was provided that applies to an array with arbitrary element beampatterns, and to near-field and reverberative propagation models. We showed that the average array DF cannot be modified by altering the positions or beampatterns of array elements. However, it is possible to achieve higher DFs for a specific set of look directions.

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