Recursive Maximum Likelihood Algorithm for Dependent Observations

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***Abstract*—A recursive maximum-likelihood algorithm (RML) is proposed that can be used when both the observations and the hidden data have continuous values and are statistically dependent between different time samples. The algorithm recursively approximates the probability density functions of the observed and hidden data by analytically computing the integrals with respect to the state variables, where the parameters are updated using gradient steps. A full convergence proof is given, based on the ordinary differential equation approach, which shows that the algorithm converges to a local minimum of the Kullback-Leibler divergence between the true and the estimated parametric probability density functions; a result which is useful even for a miss-specified parametric model. Compared to other RML algorithms proposed in the literature, this contribution extends the state-space model and provides a theoretical analysis in a non-trivial statistical model that was not analyzed so far. We further extend the RML analysis to constrained parameter estimation problems. Two examples, including nonlinear state-space models, are given to highlight this contribution.**

I. Introduction

State-space is an important model in signal processing, which includes hidden data in addition to the observation. While the likelihood of the observed data is usually very complicated, the joint distribution of the complete data is usually more tractable. An important tool is the expectation-maximization (EM) algorithm [1] which iterates between an expectation step (E-step) and a maximization step (M-step). While ensuring a monotonic increase of the likelihood the EM may be computationally expensive in online applications since it requires carrying out the iterations whenever a new sample arrives.

To solve this problem, a recursive version of the EM algorithm, based on a Newton search, was suggested by Titterington [2]. Under the assumption of independent observations, an almost surely convergence of Titterington's REM (TREM) algorithm was presented by Wang and Zhao in [3], based on the results in Delyon [4]. A stochastic approximation version for the EM algorithm was proposed by Delyon et al. in [5], and its convergence was proven therein. A further study of the recursive EM (REM) approach appears in [6] for the direction of arrival estimation using TREM, which includes another recursive algorithm suggested by the authors. It is also shown that both algorithms converge with probability one (w.p.1) to a stationary point of the likelihood function. Unlike the iterative EM that optimizes the likelihood function, the theoretical analysis of REM methods in [2]–[6] is related to a different approach usually referred to as *stochastic approximation (SA)*. Since the early 1950's and the watershed work of Robins and Monro [7], numerous recursive algorithms and their theoretical analysis have been investigated in the field of SA. One of the main methods to analyze SA algorithms is to use an ordinary differential equation (ODE) to characterize the series of estimators [8]. An introduction to the design of SA algorithms as well as their ODE-based analysis can be found in [9]–[11].

The recursive EM algorithms in [2]–[6] are suitable for online applications. However, they rely on the assumption that the observations are independent, identically distributed (i.i.d.), which invalidates the theoretical results of these algorithms when that assumption is violated.

Among the statistical models with dependency over time, the hidden Markov model (HMM) is particularly important. The HMM is a parametric model where the hidden signal is assumed to be in one of a finite number of states, and the transition between states is controlled by a matrix of probabilities. A pioneering work regarding the estimation of the HMM parameters was presented in [12], which includes several results about the maximum likelihood (ML) estimator.

The above hidden data approaches are based on either gradient or Newton steps, and aim at maximizing the likelihood function (or minimizing the Kullback-Leibler divergence (KLD)) with respect to (w.r.t.) the parameter of interest. A recursive-EM approach was proposed by Cappé and Moulines [13] for exponential-family distributions, in which the parameter and the sufficient statistic of the problem are updated simultaneously. In the E-step, the sufficient statistic is recursively updated using the latest observation, and in the M-step, the parameter is optimized accordingly. The resulting sequence of estimates was shown to converge to a local minimum of the KLD in the case of independent observations [13]. The same approach was used in [14] for HMM where both the hidden data and the observations are discrete. An extension of [14] to the case of continuous observations with discrete HMM is given in [15], and an extension to a general state-space model using sequential Monte-Carlo (SMC) can be found in [16].

In [17], a recursive-batch gradient method for HMM models was presented where each recursion processes a data-block. Two other recursive algorithms for HMM problems were proposed in [18], namely, the recursive maximum likelihood (RML) and the recursive conditional least square estimator (RCLSE). The convergence and the efficiency of these algorithms were demonstrated in [18]. A thorough theoretical asymptotic analysis of the RML for HMM is given in [19], which includes, among other issues, the convergence proof and convergence rate of the algorithm. In [20], the asymptotic bias of a stochastic gradient search is analyzed, and its application to HMM is discussed.

In the case of linear continuous state-space models, an iterative EM algorithm for the simultaneous estimation of the hidden signal

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and the parameters was proposed in [21], [22]. Two online alternatives to the EM algorithm where proposed. In the first algorithm, dubbed the recursive Kalman-EM (RKEM), the Kalman smoother is substituted by the Kalman filter such that the iterative EM is turned into an online EM algorithm. The second algorithms is the RML (though it is coined gradient based modified M-step) which is derived using a previous result [23]. The two algorithms were extended for the online speech enhancement in [24] by simultaneously estimating its autoregressive (AR) parameters and the clean speech signal.[1] Another application of the RKEM approach is presented in [25], where an online estimation of an acoustic transfer function and the clean speech signal were proposed. Titterington's and Cappé-Moulines' schemes were used for the multiple speaker tracking problem in [26], in which a constrained version for Titterington's algorithm was also proposed. In [27], the convergence rate of the batch EM algorithm was compared to three REM variants using different estimation tasks. The results show that online algorithms have a better convergence rate, and in some cases, even more accurate results.

In the case of state-space models,[2] the calculation of the RML is more involved because, in most of the cases, there is no closed-form expression for the underlying filtering problem (see [29] for an elaborated review of the topic). Possible solutions are linearization (cf. [30]) and particle filter approaches. The RML for such problems was derived in [28], where the gradient of the log-likelihood is combined with particle filtering. The latter makes the algorithm applicable even if there is no closed-form filtering formulae, which is the case in most nonlinear state-space models, a property that makes this particle filtering approach very useful [28], [31]–[34]. An important landmark in particle filter based RML for state-space models is the algorithm proposed in [35], which addresses the problem of path degeneracy and provides insights. The properties of this algorithm were further analyzed theoretically in [36], [37].

An important problem becomes apparent when extending the state-space model such that the observed signal effect the hidden state. A notable example is the Lombard effect in which speakers increase their vocal effort and shift the uttered formants when speaking in noisy environments, usually improving speech intelligibility [38]. Considering the spoken speech as the hidden state, and the acoustic signal measured at the speaker's ears as the observed signal, the Lombard effect can be analyzed by the proposed extended model.

The RML for the proposed generalized state-space model (Sec. III-A) also applies to the RML in other state-space models. The convergence of the RML in state-space models was implied in [28], but is not well covered in the literature and there is no accessible, self-contained proof.[3] A possible reason why this convergence is not well covered is that it is considered impossible to derive it without applying particle filtering methods.

In some nonlinear cases a closed-form filtering formulae exist. In initial works by Benes [41] and Mitter [42], exact, continuous-time filters were derived for nonlinear, continuous-time statistical models. These filters found interest in the problem of filtering in diffusion processes [43], [44]. In [45], the Benes filter was used as a benchmark for comparison of different nonlinear filters: the extended Kalman filter, statistical linearization and particle filtering. Different exact filters for nonlinear models were derived in [46], where a discrete-time double stochastic autoregressive process is considered.

The models that the Benes filter solves are a subclass of models that are solved by the Daum filter (DF). In [47]–[49], Daum presented a family of discrete-time, nonlinear optimal filters for continuous-time statistical models. Daum's results require that the evolution of the hidden process will satisfy the Fokker-Planck equation [50]. It is shown in [49] that under this requirement and some other regularity conditions, the conditional probability of the hidden signal given the observation has a closed-form. The applicability of the DF is demonstrated in [51], where Schmidt proposes a nonlinear filter based on Daum's theory. A nonlinear filter for the exoatmospheric intercept of an intercontinental ballistic missile is given as an example.

In this paper, we study the RML algorithm for the generalization of state-space models mentioned above, in which the state process and the observations depend on a finite set of previous observations. This model also covers other state-space models, including nonlinear cases where the RML with optimal filtering was not yet analyzed. Assuming that the parameters are time-invariant, and that the KLD between the real and parameterized conditional probability density functions (p.d.f.s) of the observations converges to an asymptotic function, we show that the RML converges to a stationary point of the likelihood function. We provide an example of the generalized state-space model, for which we derive the RML and establish its convergence. Although analytical methods from the field of SA are used in this paper, the proposed algorithm differs from the original structure of SA [7], since the step-size is constant, which allows only a weak convergence. In contrary, the original SA incorporates a decreasing step-size, and a strong convergence is usually shown. However, constant step-size is attractive from other aspects e.g. real-time applications and time-varying parameters, as discussed and analyzed in [9] (Sec. 8) and [11] (Sec. 4), and we therefore adopt this approach instead.

This paper is organized as follows. Notation, and a brief review of Titterington's REM algorithm are presented in Sec. II. The formulation of the problem and the statistical hidden model appear in Sec. III-A. The proposed method is derived in Sec. III-B, and an example is presented in Sec. V. Technical assumptions required for the convergence of the proposed method are detailed in Sec. IV, and a convergence proof is given in Sec. VI. Conclusions are drawn in Sec. VII.

[1]We adopt the term RML as used in [19], which refers to an online algorithm whose step comprises only the gradient of the log-likelihood, which is updated using the newly arrived observation. In the case of state-space models this is done by taking conditional expectation of the gradient of the complete data log likelihood while invoking the Fisher identity. Similar step is made in the TREM algorithm in [2]. However, the latter is Newton-based, comprising both the scoring function and the Fisher information matrix (FIM) matrix (instead of the actual Hessian) in every step. This comparison is further elaborated on in the body of the paper.

[2]By state-space we refer to a general model with continuous, not necessarily Gaussian, state and observation processes, where the observations are independent of each other given the state process (see e.g. [28]). These models are commonly called general state-space, but since we propose a further generalization in this paper, the term state-space is used for clarity.

[3]In [28], it is noted that the convergence of the RML in state-space models can be shown using the exponential forgetting derived in [39]. In a very recent study [37], it is indicated that the convergence of the RML with exact filter follows from the general theory of stochastic approximation (Sec. 2.3), and that many details are not well covered in the literature (Sec. 4). The convergence and convergence rate of the RML in state-space models is the subject of ongoing research [37], [40].

II. Parameter Estimation using Hidden Data

In this section, the hidden-data problem is formulated, and the recursive-EM algorithm for the i.i.d. case is reviewed.

A. Problem Formulation

Let $\mathcal{Z}_n = \{z_1, z_2, ...z_n\}$, $z_i \in \mathbb{R}^{J_z}$ be a set of observations, where J_z is the dimension of the observation vector. We assume that for every sample n there exists an absolutely continuous probability distribution of $\mathcal{Z}_n$, and therefore a corresponding p.d.f., denoted by $h_{\mathcal{Z}_n}(\mathcal{Z}_n)$. Considering parametric estimation, the estimate of $h_{\mathcal{Z}_n}(\mathcal{Z}_n)$ is an element of the parametric model $\left\{f_{\mathcal{Z}_n}(\mathcal{Z}_n;\theta) : \theta \in \mathbb{R}^{J_p}\right\}$, where J_p is the dimension of the parameters vector. We do not require that $h_{\mathcal{Z}_n}(\mathcal{Z}_n) \in \left\{f_{\mathcal{Z}_n}(\mathcal{Z}_n;\theta), \theta \in \mathbb{R}^{J_p}\right\}$; i.e., the parametric model is not necessarily correctly specified. For simplicity, we henceforth omit the subscript $\mathcal{Z}_n$ and use $h_n(\mathcal{Z}_n)$ and $f_n(\mathcal{Z}_n;\theta)$ instead of $h_{\mathcal{Z}_n}(\mathcal{Z}_n)$ and $f_{\mathcal{Z}_n}(\mathcal{Z}_n;\theta)$, respectively.

Given the observation $\mathcal{Z}_n$, the ML estimate of θ is defined as

$$\theta_n^{\text{ML}} = \underset{\theta}{\operatorname{argmax}}\, f_n(\mathcal{Z}_n;\theta) \ . \tag{1}$$

Under some regularity conditions, as $n \to \infty$, the ML estimate approaches the minimizer of the KLD w.r.t. $h_n(\mathcal{Z}_n)$ (Theorem 3.5 in [52, p. 29]), where the KLD is given by

$$\begin{aligned} K_n(\theta) &\triangleq K\left\{h_n(\mathcal{Z}_n) \mid\mid f_n(\mathcal{Z}_n;\theta)\right\} \\ &= E\left\{\log \frac{h_n(\mathcal{Z}_n)}{f_n(\mathcal{Z}_n;\theta)}\right\}, \end{aligned} \tag{2}$$

and $E\{\cdot\}$ is the expectation w.r.t. the true p.d.f. $h_n(\mathcal{Z}_n)$.

In many signal processing applications, expressing $f_n(\mathcal{Z}_n;\theta)$ in closed form is impossible; a fact that makes the derivation of the ML much more difficult. Furthermore, even if it is possible to express $f_n(\mathcal{Z}_n;\theta)$ in closed form, the resulting optimization problem may be very difficult to solve. A common approach in such cases is to define latent data and apply the EM algorithm [1], which is known to converge to a local maximum of $f_n(\mathcal{Z}_n;\theta)$. Let $\mathcal{X}_n = \{x_1, x_2, ..., x_n\}$, $x_i \in \mathbb{R}^{J_x}$ be a set of hidden data vectors with dimension J_x, and assume that there exists a parametric joint p.d.f., denoted by $f_n(\mathcal{Z}_n, \mathcal{X}_n;\theta)$. We denote the combination of the observed and hidden data $\{\mathcal{Z}_n, \mathcal{X}_n\}$ as the complete data. The EM is an iterative algorithm where each iteration consists of two steps: expectation (E-step) and maximization (M-step). In the E-step of the $(\ell+1)$-th iteration, the algorithm evaluates the following auxiliary function

$$Q(\theta|\theta_\ell) \triangleq E_{f_n}\left\{\log f_n(\mathcal{X}_n, \mathcal{Z}_n;\theta)|\mathcal{Z}_n;\theta_\ell\right\}, \tag{3a}$$

where θ_ℓ is the estimate of θ calculated at the ℓ-th iteration, and $E_{f_n}\{\cdot|\mathcal{Z}_n;\theta_\ell\}$ denotes the expectation w.r.t. the conditional p.d.f. $f_n(\mathcal{X}_n|\mathcal{Z}_n;\theta_\ell)$. Then, in the M-step, θ_ℓ is updated according to

$$\theta_{\ell+1} = \underset{\theta}{\operatorname{argmax}}\, Q(\theta|\theta_\ell) \, . \tag{3b}$$

As evident from (3a), the EM algorithm processes the entire observation set $\mathcal{Z}_n$ at every iteration, which is very demanding from a computational and storage point of view. In many applications, a recursive algorithm is necessary; that is, an algorithm in which only a single observation is processed at a time. Such an algorithm, dubbed the recursive-EM algorithm, was proposed by Titterington [2] for the i.i.d. case, and is discussed in the next section.

B. Recursive-EM for i.i.d Observations - a Review

In the case of i.i.d. observations, the log-likelihood is given by

$$\begin{aligned} \log h_n(\mathcal{Z}_n) &= \sum_{i=1}^{n} \log h(z_i) \, , \\ \log f_n(\mathcal{Z}_n;\theta) &= \sum_{i=1}^{n} \log f(z_i;\theta) \ , \end{aligned} \tag{4}$$

where $h(z)$ and $f(z;\theta)$ denote, respectively, the actual and parametric marginal p.d.f.s of a single observation. The KLD is given by

$$K_n(\theta) = n \cdot k(\theta) \quad , \quad k(\theta) \triangleq E\left\{\log \frac{h(z)}{f(z;\theta)}\right\} \ , \tag{5}$$

hence the maximization of the marginal p.d.f. $f(z;\theta)$ is asymptotically equivalent to the minimization of $\frac{1}{n}K_n(\theta)$. Titterington's REM algorithm for maximizing $f(z;\theta)$ w.r.t. θ is based on a Newton step [2],

$$\theta_n^{\text{Nwt}} = \theta_{n-1}^{\text{Nwt}} + \left[n \cdot H(z_n;\theta_{n-1}^{\text{Nwt}})\right]^{-1} s\left(z_n;\theta_{n-1}^{\text{Nwt}}\right), \tag{6}$$

for $n \geq 1$, where the *scoring function* and the *Hessian* are given, respectively, by

$$s(z;\theta) = \nabla_\theta \log f(z;\theta)\, , \tag{7}$$

$$H(z;\theta) = -\nabla_\theta^2 \log f(z;\theta) \ , \tag{8}$$

and θ_0^{Nwt} is an initial value of the parameters vector.

Using the Fisher identity (Sec. 15 of [53]), the *scoring function* is calculated using the complete data as follows,

$$s(z;\theta) = E\left\{\nabla_\theta \log f(z, x;\theta)|z;\theta\right\} \ . \tag{9}$$

For simplicity, the *Hessian* $H(z;\theta)$ in (6) is replaced by the Fisher information matrix

$$I_C(\theta) = -E\left\{\nabla_\theta^2 \log f(x, z;\theta)\right\} \ . \tag{10}$$

As a result, Titterington's REM is given by

$$\theta_n^{\text{Ti}} = \theta_{n-1}^{\text{Ti}} + \left[n \cdot I_C\left(\theta_{n-1}^{\text{Ti}}\right)\right]^{-1} s\left(z_n;\theta_{n-1}^{\text{Ti}}\right), \tag{11}$$

for $n \geq 1$, which does not require the explicit calculation of $f(z;\theta)$. Using [54], [55], Titterington showed that if θ is a scalar, θ_n^{Ti} converges to a local minimum of the KLD; a result that was extended to the vector case more than twenty years later [3] for the exponential family.

The TREM algorithm (11), which is designed for the model (4) does not take into account time dependencies between observations. This is the topic of the remainder of this paper.

III. Recursive ML for Dependent Observations

Parameter estimation from time dependent observations and hidden data was carried out in [12], [17], [18], [28], [56]–[58] and [14], [15] assuming an HMM, or using the SMC approach. In this paper, however, we focus on a different case where the hidden data are continuous, and its p.d.f. is approximated without any discretization.In this section, we present a statistical model which is a generalization of the state-space model that includes a feedback between previous observations and the current hidden state. Then, we present the RML algorithm for this model that include a simultaneous estimation of both the hidden state and the unknown parameters. The problem is formulated in Sec. III-A, the

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proposed algorithm is derived in Sec. III-B, and is illustrated in Sec. V-B.

A. Hidden-data Model of Dependent Observations

The hidden data model is described in Assumption 1:

Assumption 1. *Let $\mathcal{Z}_{n-N+1}^{n} = \{z_{n-N+1}, ..., z_n\}$; we assume that for every integer $n \geq N$,*

$$f_n(x_n|x_{n-1}, \mathcal{Z}_{n-1}; \theta) = \phi_n(x_n|x_{n-1}, \mathcal{Z}_{n-N_\phi}^{n-1}; \theta), \quad (12a)$$

$$f_n(z_n|x_n, \mathcal{Z}_{n-1}; \theta) = m_n(z_n|x_n, \mathcal{Z}_{n-N_m}^{n-1}; \theta), \quad (12b)$$

where $N_\phi, N_m \in \mathbb{N}$ are finite natural numbers, and $m_n(\cdot|\cdot, \cdot; \theta)$ and $\phi_n(\cdot|\cdot, \cdot; \theta)$ are known conditional p.d.f.s.

Note that if $\phi_n(x_n|x_{n-1}, \mathcal{Z}_{n-N_\phi}^{n-1}; \theta) = \phi_n(x_n|x_{n-1}; \theta)$ and $m_n(z_n|x_n, \mathcal{Z}_{n-N_m}^{n-1}; \theta) = m_n(z_n|x_n; \theta)$, the model (12) is an HMM with a continuous x_n. In [12], [14], [15], [17], [18], an HMM is assumed, where x_n is discrete, and in this special case, (12a) and (12b) can be substituted by $P_n^{i,j}(\theta) \triangleq \phi_n(x_n = i|x_{n-1} = j; \theta)$ and $m_n(z_n|x_n; \theta)$, respectively. A continuous x_n is treated in [16], [28], [56]–[58] based on a discretization of $\phi_n(x_n|x_{n-1}; \theta)$ using SMC, where optimality is obtained when the number of particles increases. In this paper, a different approach is introduced, which assumes that given θ, z_n, and $f_{n-1}(x_{n-1}|\mathcal{Z}_{n-1}; \theta)$, the p.d.f. $f_n(x_n|\mathcal{Z}_n; \theta)$ can be directly calculated. It is generally not possible, though, to derive the filtering p.d.f. $f_n(x_n|\mathcal{Z}_n; \theta)$ in closed-form, and in these cases one should resort to particle filtering approximation. We provide however an example in Sec. V where the filtering p.d.f. can be obtained in closed-form, and therefore the particle filtering approximation is not necessary.

As stated in Sec. II-B, in the case of i.i.d. observations (4), the maximization of the marginal p.d.f. $f(z; \theta)$ w.r.t. θ asymptotically leads to a local minimum of the KLD between the true and estimated p.d.f.s. In the case of dependent observations, $h_n(\mathcal{Z}_n)$ and $f_n(\mathcal{Z}_n; \theta)$ cannot be expressed as in (4), but rather by using the chain rule,

$$\log h_n(\mathcal{Z}_n) = \log h_1(z_1) + \sum_{i=2}^{n} \log h_i(z_i|\mathcal{Z}_{i-1}) , \quad (13a)$$

$$\log f_n(\mathcal{Z}_n; \theta) = \log f_1(z_1; \theta) + \sum_{i=2}^{n} \log f_i(z_i|\mathcal{Z}_{i-1}; \theta) , \quad (13b)$$

for every $n \geq 2$. In this paper, we use the following variant of the KLD,

$$\bar{k}_n(\theta) \triangleq E\left\{\log \frac{h_n(z_n|\mathcal{Z}_{n-1})}{f_n(z_n|\mathcal{Z}_{n-1}; \theta)}\right\} , \quad (14)$$

which satisfies

$$K_n(\theta) = K_1(\theta) + \sum_{i=2}^{n} \bar{k}_i(\theta) . \quad (15)$$

We assume the following stationarity and continuity of the KLD:

Assumption 2. *$\bar{k}_n(\theta)$ is a differentiable function of θ. Furthermore, the limits $\bar{k}_n(\theta) \xrightarrow{n\to\infty} \bar{k}(\theta)$ and $\nabla_\theta \bar{k}_n(\theta) \xrightarrow{n\to\infty} \nabla_\theta \bar{k}(\theta)$ exist uniformly in θ.*

From (15) and Asm. 2, it follows that

$$\bar{k}(\theta) - \frac{1}{n} K_n(\theta) \xrightarrow{n\to\infty} 0 . \quad (16)$$

If $\bar{k}(\theta)$ has a single minimum at $\theta = \theta^*$, then by the continuity of $\bar{k}(\theta)$, by (16), and by invoking Theorem 3.4 in [52], it follows that $(\theta_n^{\mathrm{ML}} - \theta^*) \xrightarrow{\text{w.p.1}} 0$ as $n \to \infty$. Therefore, the minimization of $\bar{k}(\theta)$ is asymptotically equivalent to the maximization of the likelihood.

Below, the minimization of $\bar{k}(\theta)$ w.r.t. θ is carried out by approximating the gradient step $\nabla_\theta \bar{k}(\theta)$ (similar to Titterington's REM (11), where an approximation of the Newton step is applied). Using a hidden data model, we calculate the derivative via the identity (proven in App. A),

$$\begin{aligned} \nabla_\theta \bar{k}(\theta) &= -\nabla_\theta E\left\{\log f_n\left(z_n|\mathcal{Z}_{n-1}; \theta\right)\right\} \\ &= -E\left\{E_{f_n}\left\{\nabla_\theta \log f_n\left(z_n, x_n|\mathcal{Z}_{n-1}; \theta\right)|\mathcal{Z}_n; \theta\right\}\right\} , \end{aligned} \quad (17)$$

where $E_{f_n}\{\cdot|\mathcal{Z}_n; \theta\}$ denotes the conditional expectation operator w.r.t. $f_n(x_n|\mathcal{Z}_n; \theta)$.

B. Proposed Algorithm

Let θ_0 be the initial estimate of θ, and consider the following gradient method for the minimization of $\bar{k}(\theta)$,

$$\theta_n = \theta_{n-1} + \epsilon \cdot Y_n(\theta_{n-1}), \quad (18a)$$

$$Y_n(\theta) = \nabla_\theta \log f_n\left(z_n|\mathcal{Z}_{n-1}; \theta\right) , \quad (18b)$$

for every $n \geq 1$, where $\epsilon > 0$ is the (small) step-size. From (17) we have

$$Y_n(\theta) = E_{f_n}\left\{\nabla_\theta \log f_n\left(z_n, x_n|\mathcal{Z}_{n-1}; \theta\right)|\mathcal{Z}_n; \theta\right\} . \quad (19)$$

The gradient $Y_n(\theta)$ substitutes the *scoring function* in Titterington's algorithm (9), but its computation is more challenging. Under Asm. 1, the conditional p.d.f. $f_n(z_n, x_n|\mathcal{Z}_{n-1}; \theta)$ can be obtained via the following recursive equations,

$$\begin{aligned} f_n\left(x_n|\mathcal{Z}_{n-1}; \theta\right) = \int & \phi_n(x_n|x_{n-1}, \mathcal{Z}_{n-N_\phi}^{n-1}; \theta) \\ & \times f_{n-1}(x_{n-1}|\mathcal{Z}_{n-1}; \theta)\, dx_{n-1}, \end{aligned} \quad (20a)$$

$$\begin{aligned} f_n\left(x_n, z_n|\mathcal{Z}_{n-1}; \theta\right) = {} & m_n(z_n|x_n, \mathcal{Z}_{n-N_m}^{n-1}; \theta) \\ & \times f_n(x_n|\mathcal{Z}_{n-1}; \theta) , \end{aligned} \quad (20b)$$

$$f_n(z_n|\mathcal{Z}_{n-1}; \theta) = \int f_n\left(x_n, z_n|\mathcal{Z}_{n-1}; \theta\right)\, dx_n , \quad (20c)$$

$$f_n(x_n|\mathcal{Z}_n; \theta) = \frac{f_n\left(x_n, z_n|\mathcal{Z}_{n-1}; \theta\right)}{f_n(z_n|\mathcal{Z}_{n-1}; \theta)} , \quad (20d)$$

and (20b) is the specific term used to calculate $Y_n(\theta)$ in (19). Under a discrete HMM model, the integrals in (20) can be substituted by matrix multiplications, and in the Gaussian case, by the Kalman filtering formulae (cf. [59], Sec. 3.1). In other cases, the integrals in (20) cannot be computed analytically, and SMC may be used, as proposed in [16], [28], [56]–[58]. In this paper, we focus on the cases where the integral can be computed analytically, which is more accurate than SMC approximation.

The gradient method described in (18)–(20) cannot, however, be straightforwardly used online; i.e., updating θ_{n-1} by (18a) is not solely a function of z_n and θ_n, but requires the entire observation set $\mathcal{Z}_n$. To see this, consider the $(n-1)$-th time instance where θ_{n-1} is available, based on all the previous observations in $\mathcal{Z}_{n-1}$. Now assume that an additional observation z_n is now acquired, and that one wishes to update the estimate θ_n according to (18a). To do so, one must calculate the gradient in (17) by substituting $\theta = \theta_{n-1}$, which requires a recursive application of (20). This calculation involves the entire observation set $\mathcal{Z}_n$, since one must substitute $\theta = \theta_{n-1}$ in every step, which was not possible previously. After θ_n was obtained, when the next measurement z_{n+1} is acquired, obtaining θ_{n+1} requires repeating the recursion all over again using the entire measurement set $\mathcal{Z}_{n+1}$. Note that besides maintaining the entire data set, the computational complexity of such an algorithm is exponential with n.

To derive an online algorithm, we replace the p.d.f.s in (20) with an approximation where θ_{n-1} is substituted in (20a),

$$f_n^\epsilon(x_n|\mathcal{Z}_{n-1};\theta) = \int \phi_n(x_n|x_{n-1},\mathcal{Z}_{n-N_\phi}^{n-1};\theta) \times f_{n-1}^\epsilon(x_{n-1}|\mathcal{Z}_{n-1};\theta_{n-1}^\epsilon)\, dx_{n-1} , \tag{21a}$$

$$f_n^\epsilon(x_n,z_n|\mathcal{Z}_{n-1};\theta) = m_n(z_n|x_n,\mathcal{Z}_{n-N_m}^{n-1};\theta) \times f_n^\epsilon(x_n|\mathcal{Z}_{n-1};\theta) , \tag{21b}$$

$$f_n^\epsilon(z_n|\mathcal{Z}_{n-1};\theta) = \int f_n^\epsilon(x_n,z_n|\mathcal{Z}_{n-1};\theta)\, dx_n , \tag{21c}$$

$$f_n^\epsilon(x_n|\mathcal{Z}_n;\theta) = \frac{f_n^\epsilon(x_n,z_n|\mathcal{Z}_{n-1};\theta)}{f_n^\epsilon(z_n|\mathcal{Z}_{n-1};\theta)} , \tag{21d}$$

where θ_{n-1}^ϵ is the estimator of θ from iteration $n-1$ based on $\mathcal{Z}_{n-1}$. The crucial difference between (20) and (21) is the use of $f_{n-1}^\epsilon(x_{n-1}|\mathcal{Z}_{n-1};\theta_{n-1}^\epsilon)$ in (21a) which is determined by $\mathcal{Z}_{n-1}$ and does not comprise θ as a parameter, unlike $f_{n-1}(x_{n-1}|\mathcal{Z}_{n-1};\theta)$ in (20a). Thus, when the estimate of θ is updated, (21) does not repeat the recursion from $n = 1$.

Now that we have substituted (20) by (21a)–(21d), an online algorithm can now be derived by substituting the resulting p.d.f.s into (18), i.e.,

$$Y_n^\epsilon(\theta) = E_n^\epsilon\left\{\nabla_\theta \log f_n^\epsilon(z_n,x_n|\mathcal{Z}_{n-1};\theta)|\mathcal{Z}_n\right\} , \tag{21e}$$

where $E_n^\epsilon\{\cdot|\mathcal{Z}_n\}$ denotes the expectation taken w.r.t. $f_n^\epsilon(x_n|\mathcal{Z}_n;\theta_{n-1}^\epsilon)$. Finally, the update previously given in (18), is now

$$\theta_n^\epsilon = \theta_{n-1}^\epsilon + \epsilon \cdot Y_n^\epsilon(\theta_{n-1}^\epsilon) . \tag{21f}$$

The proposed RML algorithm now has the general structure of an EM algorithm, where in the E-step the p.d.f.s are propagated, and in the M-step, the parameter update is calculated. The algorithm is summarized in Alg. 1.

Comment 1. *When the observations are independent,* $f_n^\epsilon(z_n,x_n|\mathcal{Z}_{n-1};\theta)$ *is equal to* $f_n(z_n,x_n;\theta)$ *and* (21e) *boils down to* $E_n\{\nabla_\theta \log f_n(z_n,x_n;\theta)|z_n\}|_{\theta=\theta_{n-1}^\epsilon}$*, which is equal to* $s(z_n;\theta_n)$ *in* (11)*. Note that in the i.i.d. case, the recursive calculation of the p.d.f.s* (21a) – (21d) *requires only the computation of* $s(z_n;\theta_n)$*. In fact, the proposed algorithm can be seen as a generalization of Titterington's REM to the non-i.i.d. case, and using only a gradient step instead of a Newton step. An extension of* (21) *that uses a Newton step is apposite, but is beyond the scope of this paper.*

Comment 2. *The general structure of* (21) *is similar to [17], [18], [28], [56]–[58], but the main difference is in the statistical model, and the approximation of the p.d.f.. Whereas a discrete HMM was assumed in [17], [18], a more general model is assumed in [28], [56]–[58], and Monte-Carlo methods are therefore required. In this light, the algorithm in* (21) *is useful for dependent observations with continuous-domain hidden data, where the propagation of the p.d.f.s* (21a) – (21d) *can be analytically calculated, and discrete (HMM or particle filtering) approximations are not needed.*

Algorithm 1: The proposed approach.

E-step:

(21a) $f_n^\epsilon(x_n|\mathcal{Z}_{n-1};\theta) = \int \phi_n(x_n|x_{n-1},\mathcal{Z}_{n-N_\phi}^{n-1};\theta) \times f_{n-1}^\epsilon(x_{n-1}|\mathcal{Z}_{n-1};\theta_{n-1}^\epsilon)\, dx_{n-1}$

(21b) $f_n^\epsilon(x_n,z_n|\mathcal{Z}_{n-1};\theta) = m_n(z_n|x_n,\mathcal{Z}_{n-N_m}^{n-1};\theta) \times f_n^\epsilon(x_n|\mathcal{Z}_{n-1};\theta)$

(21c) $f_n^\epsilon(z_n|\mathcal{Z}_{n-1};\theta) = \int f_n^\epsilon(x_n,z_n|\mathcal{Z}_{n-1};\theta)\, dx_n$

(21d) $f_n^\epsilon(x_n|\mathcal{Z}_n;\theta) = \dfrac{f_n^\epsilon(x_n,z_n|\mathcal{Z}_{n-1};\theta)}{f_n^\epsilon(z_n|\mathcal{Z}_{n-1};\theta)}$

M-step:

(21e) $Y_n^\epsilon(\theta) = E_n^\epsilon\{\nabla_\theta \log f_n^\epsilon(z_n,x_n|\mathcal{Z}_{n-1};\theta)|\mathcal{Z}_n\}$

(21f) $\theta_n^\epsilon = \theta_{n-1}^\epsilon + \epsilon \cdot Y_n^\epsilon(\theta_{n-1}^\epsilon)$

IV. Convergence of the Proposed Algorithm

In this section we present the main theorem, a convergence theorem for the proposed algorithm, and discuss the necessary regularity conditions. To further understand the regularity conditions, we conclude this section by showing that they are satisfied in the example given in Sec. V-B.

A. Regularity Conditions

Assumption 3. *The function* $f_n^\epsilon(x_n|\mathcal{Z}_n;\theta)$ *defined in* (21d) *is a continuous function of* θ*, uniformly in* n *and* θ*, w.p.1.*

Assumption 4. $\log f_n(z_n|\mathcal{Z}_{n-1};\theta)$ *is continuously differentiable w.r.t.* θ*, uniformly in* n *and* θ*, w.p.1.*

Assumption 5. *Let* $\nu' \triangleq \nu'(x_m|\mathcal{Z}_m)$ *and* $\nu'' \triangleq \nu''(x_m|\mathcal{Z}_m)$ *be conditional p.d.f.s. For every* $n > m$*, let* $f_n'(z_n|\mathcal{Z}_{n-1};\theta)$ *and* $f_n''(z_n|\mathcal{Z}_{n-1};\theta)$ *be the conditional p.d.f.s obtained by applying* (20) *sequentially from time instant* $m+1$ *to time instant* n*, where* $f_m'(x_m|\mathcal{Z}_m;\theta) = \nu'(x_m|\mathcal{Z}_m)$*, and respectively* $f_m''(x_m|\mathcal{Z}_m;\theta) = \nu''(x_m|\mathcal{Z}_m)$*, were used as the initial p.d.f.s at time* m*. We assume that,*

$$\nabla_\theta \bar{k}'(\theta) - \nabla_\theta \bar{k}''(\theta) \xrightarrow{n\to\infty} 0 , \tag{22}$$

uniformly in θ *where*

$$\bar{k}_n'(\theta) \equiv E\left\{\log \frac{h_n(z_n|\mathcal{Z}_{n-1};\theta)}{f_n'(z_n|\mathcal{Z}_{n-1};\theta)}\right\} ,$$

$$\bar{k}_n''(\theta) \equiv E\left\{\log \frac{h_n(z_n|\mathcal{Z}_{n-1};\theta)}{f_n''(z_n|\mathcal{Z}_{n-1};\theta)}\right\} .$$

Asm. 5 simply implies that the initial p.d.f. does not affect the asymptotic gradient steps. Note that this assumption can be established in general state-space models under some regularity conditions [39]. Asms. 6–8 below are required for the calculation of the gradient in (21e):

Assumption 6. *Both* $\phi_n(\cdot|\cdot,\cdot\,\theta)$ *and* $m_n(\cdot|\cdot,\cdot\,;\theta)$ *are uniformly continuous functions of* θ*, uniformly*[4] *in* n*, w.p.1.*

Assumption 7. Y_n^ϵ *is integrable w.r.t. the p.d.f.* $h_n(\mathcal{Z}_n)$*, uniformly*[5] *in* n *and* ϵ*.*

[4] see e.g., [60] for details.
[5] see [61, p. 65] for details.

Assumption 8. *For every $\theta \in \Theta$,*

$$E\left\{\nabla_\theta \log f_n\left(z_n|\mathcal{Z}_{n-1};\theta\right)\right\} \\ = \nabla_\theta E\left\{\log f_n\left(z_n|\mathcal{Z}_{n-1};\theta\right)\right\} \ . \quad (23)$$

Note that the right-hand side of (23) is equal to $-\nabla_\theta \bar{k}_n(\theta)$ (and $\bar{k}_n(\theta)$ is defined in (14)). Sufficient conditions to satisfy Asm. 8 can be found by using Theorem 2.27 in [60, p. 56].

B. Convergence Theorem

The convergence of Alg. 1 can be shown by using the ODE method for stochastic approximation [9], which analyzes the stochastic sequence θ_n^ϵ by looking at a continuous stochastic process $\theta^\epsilon(t)$, which is an interpolation of $\{\theta_n^\epsilon\}$. The idea is to show that as $\epsilon \to 0$, the interpolated process $\theta^\epsilon(t)$ converges to a process $\theta(t)$ whose sample paths are solutions of a differential equation which characterizes the asymptotic behavior of the algorithm.

Using (21f), we define the following process, which is similar to the zero-order hold (ZOH) interpolation,

$$\theta^\epsilon(t) \triangleq \begin{cases} \theta_0 & ; \quad t < 0 \\ \theta_n^\epsilon & ; \quad t \in [n\epsilon, n\epsilon + \epsilon) \ , \ n \geq 0 \ , \end{cases} \quad (24a)$$

which can be written by explicitly applying the recursion

$$\theta^\epsilon(t) = \theta_0 + \epsilon \cdot \sum_{i=1}^{\lfloor t/\epsilon \rfloor} Y_i^\epsilon \ , \ t \geq 0 \ . \quad (24b)$$

Note that determining $\theta^\epsilon(t)$ for every $t \geq 0$, also determines θ_n^ϵ for every n.

We are now ready to present the main theorem,

Theorem 9. *Consider $\theta^\epsilon(t)$ defined in (24a). Then, under assumptions 2–8, $\theta^\epsilon(t) \xrightarrow{D} \theta(t)$ as $\epsilon \to 0$, where $\theta(t)$ is a stochastic process such that $\bar{k}\left[\theta(t)\right]$ is monotonically decreasing in t, with probability one (w.p.1.).*

Proof. See Section VI. □

Note that $\bar{k}\left[\theta(t)\right]$ is non-negative by definition; hence $\theta(t)$ converges to a local minimum. Since Theorem 9 states the convergence properties of the interpolated process $\theta^\epsilon(t)$, the exact implication of this theorem on the actual series of estimates, $\{\theta_n^\epsilon\}$ is required. An analysis of θ_n^ϵ as n and ϵ approach infinity and zero, respectively, is based on conventional tools from large deviation theory, which are beyond the scope of this paper. Roughly speaking, the chances are very small that once θ_n^ϵ enters a small neighborhood of a minimum of $\bar{k}(\theta)$, it will escape this neighborhood. A detailed discussion of this topic is available in [9, p. 249].

The next corollary extends Theorem 9 to the case of a constrained parameter set,

Corollary 10. *Let $\theta^{(j)}$ denote the j-th element in θ, and define the constraint subset $H \subseteq \mathbb{R}^{J_p}$,*

$$H = \left\{\theta \in \mathbb{R}^{J_p} : \ a_j \leq \theta^{(j)} \leq b_j \ , \ j = 1, ..., J_p\right\} \ , \quad (25a)$$

and the corresponding constrained projection operator,

$$\Pi_H(\theta) = \underset{\theta'}{\operatorname{argmin}}\left\{\|\theta' - \theta\| : \theta' \in H\right\} \ . \quad (25b)$$

Consider the process $\theta^\epsilon(t)$ in (24a) *where θ_n^ϵ is calculated using the same procedure as in* (21)*, except for* (21f) *which is substituted by*

$$\theta_{n+1}^\epsilon = \Pi_H\left(\theta_n^\epsilon + \epsilon \cdot Y_n^\epsilon\right) \ . \quad (26)$$

Then, under Assumptions 2–8, $\theta^\epsilon(t) \xrightarrow{D} \dot{\theta}(t)$ as $\epsilon \to 0$, where $\dot{\theta}(t) \in H$ is a stochastic process such that $\bar{k}\left[\dot{\theta}(t)\right]$ is monotonically decreasing in t.

Proof. See App. C □

For other types of constraints which are commonly applied in stochastic approximation, the reader is referred to [9], Sec. 4.3.

Comment 3. *In this paper, we chose to use a constant step-size, which enables only a weak convergence proof and suffers from a systematic bias. However, using constant step-size is attractive from other aspects e.g. in real-time applications and in tracking time-varying parameters, as discussed and analyzed in [9] (Sec. 8) and [11] (Sec. 4), and we therefore adopt this approach instead.*

V. Examples

In this section, we survey several estimation problems which can benefit from the method proposed in this paper, and provide a 6 In Sec. V-A, we discuss a model where the past observations are fed back to the hidden state probability function, a feedback that was not yet addressed in the literature. In Sec. V-B, the well-known linear-Gaussian model is briefly discussed, showing the difference between the recursive prediction-error method (RPEM) algorithm discussed in [10] and the proposed algorithm. To better understand the contribution of this paper, each example is preceded by a short survey of closely-related methods and statistical models.

A. Autoregressive non-Gaussian Model

In this section we discuss a problem with a statistical model for which, to the best of our knowledge, a recursive algorithm was not derived. This problem extends the AR non-Gaussian models, by feeding back previous observations into the hidden state in addition to feeding it into the current observation. Before we discuss this example in detail, we begin with a literature survey of AR non-Gaussian models.

An important class of AR processes is the Markov switching model that was introduced in [62] for econometric data. Due to the success of this model in econometric research, the model was further investigated, and the asymptotics of estimators in Markov-switching autoregressions were investigated in [63]–[65]. While the RML for HMM models was derived in [18] and analytically discussed, this result was extended in [66] to AR model with underlying HMM process. Further details can be found in [67].

A different model used in econometrics is the autoregressive conditional heteroscedasticity (ARCH) first proposed in [68] for the estimation of inflation variance. Unlike the previous models, the hidden process is of a continuous support, and its previous samples affect the variance of the current sample. In other words, ARCH model contains an AR model for variances. An extended model, namely the generalized autoregressive conditional heteroscedasticity (GARCH), was also proposed and analytically investigated in [69]–[71]. Application of the GARCH model to speech processing was proposed in [72]–[74]. In [74], an RML algorithm to recursively estimate the model parameters is derived, but a full proof of convergence is not given. Since the hidden process in GARCH is of a continuous range, the convergence proof in [66] is not suitable. To the best of our knowledge, convergence proof for the GARCH model was not given in the literature, while this result can be deduced using the proof in this paper.

Other non-Gaussian AR models were used in different signal processing publications. In [75], a statistical model is derived, where the hidden signal is a Gaussian mixture model (GMM) independent along the time axis, and the observations are modelled as an AR process with the hidden signal as an excitation. A generalized EM (GEM) algorithm is then derived for this model. Although relatively simple, this is a useful model in several signal processing applications, as discussed in the following. The model in [75] was extended in [76] for multichannel source separation. In the field of speech processing, using an AR system to model the observations was used in many papers dealing with speech dereverberation, e.g. in [77]–[79], while the GMM model was used for speech separation [80]. In fact, both speech separation with GMM model and dereverberation with AR model were used in a practical method in [78].

In this section, we present a simple AR model which is nevertheless not covered by any of the papers in the above. As in other models, the observations are modelled as an AR process, and the hidden signal is a GMM process whose p.d.f., unlike the other models, depends on the previous observation. This model might be useful where the signal of interest is affected by the observed signal, such as in e.g. the Lombard effect in which speakers increase their vocal effort and shift their formants when speaking in noisy environments, resulting in a higher signal-to-noise ratio and better intelligibility [38].

1) Statistical Model and Filtering: The observations are given by

$$z_n = \alpha \cdot z_{n-1} + y_n \tag{27}$$

where $|\alpha| < 1$ and given x_n, y_n is an i.i.d. GMM process,

$$y_n = \sigma(x_n) \cdot u_n + \mu(x_n) \quad , \quad u_n \sim \mathcal{N}(0,1) \ , \tag{28}$$

where x_n is a discrete state taking its values in $\{0, ..., N_x - 1\}$, and $\mu(x_n), \sigma^2(x_n)$ the respective mean and variance of y_n. The state depends on the previous sample by

$$\Pr\{x_n = x | \mathcal{Z}_{n-1}; \theta\} = \begin{cases} q(x) & ; \quad |z_{n-1}| < \xi_{\text{thr}} \\ p(x) & ; \quad \text{else} \end{cases} \ , \tag{29}$$

where $p(x)$ represents the probability function of x if the previous sample z_{n-1} exceeds a given threshold ξ_{th}, whereas $q(x)$ is the probability of x otherwise. In the following we will use for brevity

$$\Pr\{x_n = x | \mathcal{Z}_{n-1}; \theta\} \equiv \phi(x | z_{n-1}; \theta) \ . \tag{30}$$

In this case, the propagation and observation p.d.f.s (12a)–(12b) are

$$\phi(x_n = x | x_{n-1}, \mathcal{Z}_{n-N_\phi}^{n-1}; \theta) = \phi(x | z_{n-1}; \theta) \ , \tag{31a}$$

$$\begin{aligned} m(z_n | x_n, \mathcal{Z}_{n-N_m}^{n-1}; \theta) &= m(z_n | x_n, z_{n-1}; \theta) \\ &= \mathcal{N}\left(z_n - \alpha \cdot z_{n-1}; \mu(x_n), \sigma^2(x_n)\right) \ . \end{aligned} \tag{31b}$$

The filtering formulae are therefore given by

$$f(x_n | \mathcal{Z}_n; \theta) = \frac{f(x_n, z_n | \mathcal{Z}_{n-1}; \theta)}{f(z_n | \mathcal{Z}_{n-1}; \theta)} \tag{32a}$$

$$\begin{aligned} f(x_n, z_n | \mathcal{Z}_{n-1}; \theta) &= \phi(x_n | z_{n-1}; \theta) \\ &\times \mathcal{N}\left(z_n - \alpha \cdot z_{n-1}; \mu(x_n), \sigma^2(x_n)\right) \end{aligned} \tag{32b}$$

$$f(z_n | \mathcal{Z}_{n-1}; \theta) = \sum_{x_n=0}^{N_x-1} f(x_n, z_n | \mathcal{Z}_{n-1}; \theta) \ . \tag{32c}$$

where

$$\theta = \{\alpha, \mu(x), \sigma(x), q(x), p(x) \ : \ x = 0, ..., N_x - 1\} \ . \tag{33}$$

For brevity, we denote $w_{n,x} = f(x_n = x | \mathcal{Z}_n; \theta)$.

2) Realization of the RML algorithm: The conditional log-likelihood in this case is

$$\begin{aligned} \log f(x_n, z_n | \mathcal{Z}_{n-1}; \theta) &= \log \phi(x_n | z_{n-1}; \theta) - 0.5 \log \sigma^2(x_n) \\ &- 0.5\sigma^{-2}(x_n) \cdot (z_n - \alpha \cdot z_{n-1} - \mu(x_n))^2 \ . \end{aligned} \tag{34}$$

The gradient steps of the RML algorithm are therefore,

$$Y_n^{\mu(x)} = w_{n,x} \cdot \sigma^{-2}(j) \cdot e_{n,x} \ , \tag{35a}$$

$$Y_n^{\sigma^2(x)} = w_{n,x} \cdot \left[-0.5 \cdot \sigma^2(x) + 0.5 \cdot \sigma^{-4}(x) \cdot e_{n,x}^2\right] \ , \tag{35b}$$

$$\tag{35c}$$

$$Y_n^{q(x)} = w_{n,x} / q(x) \quad ; \quad |z_{n-1}| < \xi_{\text{thr}} \tag{35d}$$

$$Y_n^{p(x)} = w_{n,x} / p(x) \quad ; \quad |z_{n-1}| \geq \xi_{\text{thr}} \tag{35e}$$

$$Y_n^{\alpha} = \bar{\sigma}_n^{-2} \cdot (z_n - \alpha \cdot z_{n-1} - \bar{\mu}_n) \cdot z_{n-1} \ . \tag{35f}$$

where $e_{n,x}$ denotes the error w.r.t. the x-th hypothesis defined by

$$e_{n,x} = z_n - \alpha \cdot z_{n-1} - \mu(x) \ , \tag{36}$$

and the hypothesis-averaged values of μ and σ^{-2} are

$$\bar{\mu}_n = \sum_{x=0}^{N_x-1} w_{n,x} \cdot \mu(x) \ , \ \bar{\sigma}_n^{-2} = \sum_{x=0}^{N_x-1} w_{n,x} \cdot \sigma^{-2}(x) \ . \tag{37a}$$

3) Convergence: In the following, we assume that $\sigma^2(x)$ and α are constrained by some predefined values, i.e., $\sigma^2(x) > \sigma^2_{v,\text{min}}$ and $|\alpha| < \alpha_{\text{max}} < 1$, a constraint that does not change the convergence properties of the algorithm, using Corollary 10. The propagation and observation p.d.f.s are given in (31a) and (31b), hence Asm. 1 is satisfied. These functions in θ are uniformly continuous functions of the parameters, as required in Asm. 6. By $|\alpha| < 1$, the conditional p.d.f. of the observation is given by

$$\begin{aligned} f(z_n | \mathcal{Z}_{n-1}; \theta) = \sum_{x=0}^{N_x-1} & \phi(x | z_{n-1}; \theta) \\ & \times \mathcal{N}\left(z_n - \alpha \cdot z_{n-1}; \mu(x), \sigma^2(x)\right) \ . \end{aligned} \tag{38}$$

Now, since $\phi(x|z;\theta)$ is not a function of n and assuming there is no mismodelling, i.e. $h_n(z_n | \mathcal{Z}_{n-1}) = f_n(z_n | \mathcal{Z}_{n-1}; \theta^*)$, Asm. 2 is satisfied. Therefore, $\log f(z_n | \mathcal{Z}_{n-1}; \theta)$ is a continuously differentiable function of the parameters, as required in Asm. 4, and the gradient and integral exchange that is required in Asm. 8 is also allowed. Assuming no mismodelling, these apply also to $h(z_n | \mathcal{Z}_{n-1})$, and combining with the previous argument, Asm. 5 immediately follows. From (32a) (recalling the constraints on α and $\sigma(x)$) it follows that $f(x_n | \mathcal{Z}_n; \theta)$ is uniformly continuous function of θ, satisfying Asm. 3. Finally, the terms for the gradient

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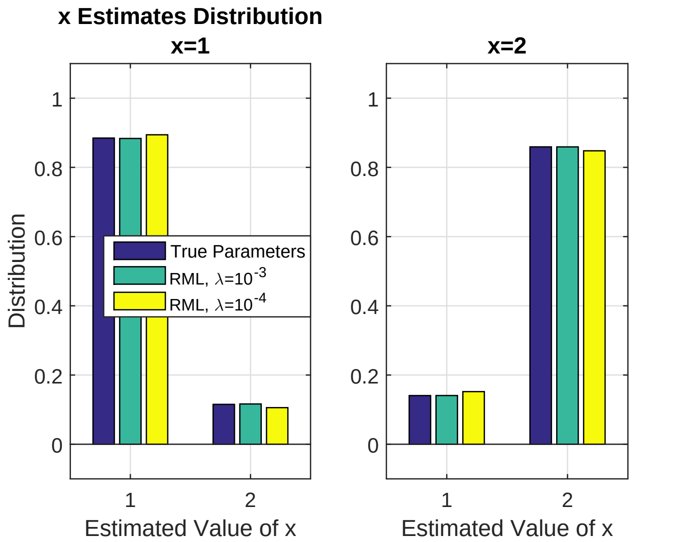

Figure 1. Numerical simulation of the non-Gaussian AR example, state estimates with two values of ϵ.

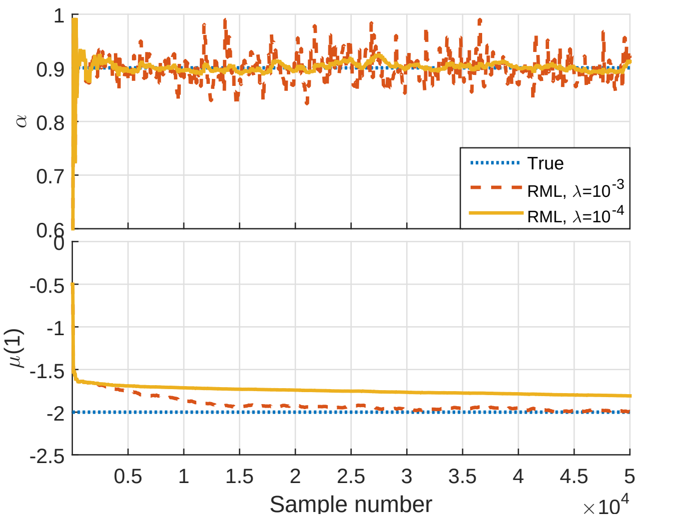

Figure 2. Numerical simulation of the non-Gaussian AR example, parameter estimates with two values of ϵ.

steps are given in (35), which includes the random variables z_n, z_{n-1} and $w_{n,x}$, which are all integrable, satisfying Asm. 7.

The proposed algorithm was numerically tested for a special case, where $N_x = 2$, $\sigma^2(x)$, $p(x)$, and $q(x)$ are known. The parameters α, $\mu(x)$ and the hidden process x_n are simultaneously estimated. The true parameters are $\alpha = 0.9$, $\mu(1) = -2$ and $\mu(2) = 2$, and the examined step-sizes are either $\epsilon = 10^{-3}$ or $\epsilon = 10^{-4}$. The state estimates using the true parameters and the RML are compared in Fig. 1, and the parameter estimates are depicted in Fig. 2. As expected, the convergence with higher step size is faster, but results in larger bias.

B. *Linear-Gaussian Model*

In this section, we discuss the application of the RML algorithm to the linear-Gaussian state-space (LGSS) model. Since its first introduction in the 1960's, the Kalman filter was extensively applied to the LGSS model. The problem of the simultaneous estimation of the state and parameters in LGSS model was later addressed in [81]–[83]. In [81], a generalized least-squares method for the estimation of parameters in autoregressive moving-average (ARMA) model is derived, and a recursive method is then presented. In [83], a single-input single-output (SISO) system is presented in an autoregressive moving-average with exogenous inputs (ARMAX) structure, and assuming an additive white Gaussian noise. To optimize the parameters, the squared prediction errors are minimized using a recursive Newton method.

The first to present analytical results for this approach was Ljung [10], [84], [85]. In [84], the strong convergence of a Landau's recursive algorithm is investigated under an ARMAX model. In Sec. IV therein, an extension of the method to the state-space model is given. The most comprehensive source for the RML algorithm for the LGSS model is the book by Ljung and Söderström [10] where many approaches for this case are presented and theoretically analyzed. However, the approach adopted in [10] is the RPEM algorithm, and not the maximization of the likelihood function. It is shown in [10] that the minimization of the prediction error is equivalent to the maximization of the likelihood, but the algorithm itself is different than the one presented in this paper.[6]

In Sec. 3.8 of [10], the RPEM algorithm for the LGSS model in its most general version is derived, and later theoretically analyzed. Two algorithms are then described which use Kalman filtering to estimate the signal and a gradient step to optimize the model parameters. In Sec. 3.8.2. of [10], the algorithm is based on the asymptotic solution of the Riccati equations derived for the covariance matrix of the Kalman filter. In other words, the asymptotic Kalman filter is used instead of the standard Kalman filter. In contrast, in Sec. 3.8.3. of [10], the algorithm uses the standard Kalman filter but, as mentioned before, the prediction error is minimized, rather than maximizing the likelihood.

To the best of our knowledge, and relying on the above survey, the RML algorithm for the LGSS model was not yet presented in the literature. In the rest of this section, we present a special case of an LGSS model and present the differences between the algorithm in Sec. 3.8.3 of [10] and the RML algorithm discussed in this paper.

1) Statistical Model: In this section, we apply the proposed algorithm to a simple hidden-data problem where the observations are dependent. Consider the following state-space,

$$x_n = \Phi \cdot x_{n-1} + w_n \quad , \quad x_n = [u_{n-1}, u_n]^\top \quad , \quad w_n = [0, u_n]^\top \ ,$$

where the propagation matrix Φ and the covariance matrix Q are given by

$$\Phi = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad , \quad Q = E\left\{w_n \cdot w_n^\top\right\} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \ . \tag{39}$$

The observed data model is

$$z_n = a \cdot x_n + v_n \quad , \quad a = [\alpha, 1] \ , \tag{40}$$

where we assume $\alpha < 1$ and that u_n and v_n are mutually uncorrelated, Gaussian, i.i.d. over n, and Gaussian with the p.d.f.s

$$u_n \sim \mathcal{N}(0, 1) \quad , \quad v_n \sim \mathcal{N}(0, \zeta_v^2) \ . \tag{41}$$

[6] Ljung's book [10] is referred to also in [11], [28] as the source for the RML approach, despite the differences mentioned now. In p. 26 of [10], the sources [86] and an earlier version of [87] are referred to as the initial sources for the RML, but these works were not published.

Assuming that neither x_n nor $\theta \equiv \{\alpha, \zeta_v^2\}$ are known, we apply the proposed RML algorithm to estimate θ. In practice, the RML recursively estimates both x_n and θ from z_n, but formally, x_n is only a byproduct of the procedure. Note that in this simple case the observations are not i.i.d., and the TREM algorithm (11) is not suitable.

2) The Proposed Algorithm: Since this model was discussed in the literature, it is here only briefly discussed. For full derivation as well as convergence proof see a supplementary material.

The filtering in this case is carried out by the following Kalman filter,

$$\widehat{x}_{n|n} = \frac{1}{\xi_n^2}\begin{bmatrix}(1+\zeta_v^2)\cdot\widehat{u}_{n-1|n-1} + \alpha\cdot\widehat{p}_{n-1|n-1}\cdot z_n \\ -\alpha\cdot\widehat{u}_{n-1|n-1} + z_n\end{bmatrix} , \quad (42a)$$

$$\widehat{P}_{n|n} = \frac{1}{\xi_n^2}\begin{bmatrix}\widehat{p}_{n-1|n-1}\cdot(1+\zeta_v^2) & -\alpha\cdot\widehat{p}_{n-1|n-1} \\ -\alpha\cdot\widehat{p}_{n-1|n-1} & \alpha^2\cdot\widehat{p}_{n-1|n-1}+\zeta_v^2\end{bmatrix} , \quad (42b)$$

where the previous state is

$$\widehat{x}_{n-1|n-1} = \begin{bmatrix}\widehat{u}_{n-2|n-1} & \widehat{u}_{n-1|n-1}\end{bmatrix}^\top , \quad (42c)$$

$$\widehat{P}_{n-1|n-1} = \begin{bmatrix}\widehat{p}_{n-2|n-1} & \widehat{p}_{n-1,n-2|n-1} \\ \widehat{p}_{n-1,n-2|n-1} & \widehat{p}_{n-1|n-1}\end{bmatrix} . \quad (42d)$$

The gradient steps in this case are

$$Y_{n,\alpha}^{\epsilon} = \zeta_v^{-2}\cdot\left(z_n\widehat{u}_{n-1|n}^{\epsilon} - \widehat{u_n u_{n-1}}^{\epsilon} - \alpha\cdot\widehat{u_{n-1}^2}^{\epsilon}\right)\Big|_{\theta=\theta_{n-1}^{\epsilon}} , \quad (43a)$$

$$Y_{n,\zeta_v^2}^{\epsilon} = -\frac{1}{2\cdot\zeta_v^2} + \frac{1}{2\cdot\zeta_v^4}\cdot\Big[z_n^2 - 2z_n\widehat{u}_{n|n}^{\epsilon} - 2\alpha z_n\widehat{u}_{n-1|n}^{\epsilon} + \widehat{u_n^2}^{\epsilon} + \alpha^2\widehat{u_{n-1}^2}^{\epsilon} + 2\alpha\widehat{u_n u_{n-1}}^{\epsilon}\Big]_{\theta=\theta_{n-1}^{\epsilon}} , \quad (43b)$$

where the first-order estimates, $\widehat{u}_{n|n}^{\epsilon}$ and $\widehat{u}_{n-1|n}^{\epsilon}$, are elements of $\widehat{x}_{n|n}^{\epsilon}$ (42a), and the second-order elements are given by

$$\widehat{u_n^2}^{\epsilon} = \widehat{u}_{n|n}^{\epsilon,2} + \widehat{p}_{n|n}^{\epsilon} , \quad (43c)$$

$$\widehat{u_{n-1}^2}^{\epsilon} = \widehat{u}_{n-1|n}^{\epsilon,2} + \widehat{p}_{n-1|n}^{\epsilon} , \quad (43d)$$

$$\widehat{u_n u_{n-1}}^{\epsilon} = \widehat{u}_{n-1|n}^{\epsilon}\cdot\widehat{u}_{n|n}^{\epsilon} + \widehat{p}_{n,n-1|n}^{\epsilon} , \quad (43e)$$

and $\widehat{p}_{n-1|n}^{\epsilon}$, $\widehat{p}_{n,n-1|n}^{\epsilon}$, and $\widehat{p}_{n|n}^{\epsilon}$ are elements of $\widehat{P}_{n|n}^{\epsilon}$ (42b).

Fig. 3 depicts the simulation results for the estimation of α and ζ_v^2 using the RML algorithm as given above for different values of ϵ. As expected, for larger ϵ, the convergence is faster, whereas the accuracy is lower, and vice versa. Many practical improvements may be considered to improve performance; for example, averaging several values of $Y_{n,\alpha}^{\epsilon}$ (or $Y_{n,\zeta_v^2}^{\epsilon}$) before updating the value of α (or ζ_v^2), or using a higher value of ϵ at the beginning for fast convergence, and after a while, decreasing ϵ for better accuracy. These considerations, however, are beyond the scope of this paper.

3) Comparison to the RPEM: The RPEM algorithm proposed in [10] fits the statistical model in Sec. V-B, and bears many similarities to the algorithm proposed in Sec. V-B2. However, these algorithms are not identical, and therefore are compared in the following. According to [10], the estimation of the model parameters is obtained by minimizing the prediction error of the estimated model defined as

$$\epsilon_n = z_n - \alpha\cdot\widehat{u}_{n-1|n-1} . \quad (44)$$

The parameter is updated by calculating the first-order derivative of (44) $\nabla_\theta\epsilon_n$. It can be seen that while the optimization of α is

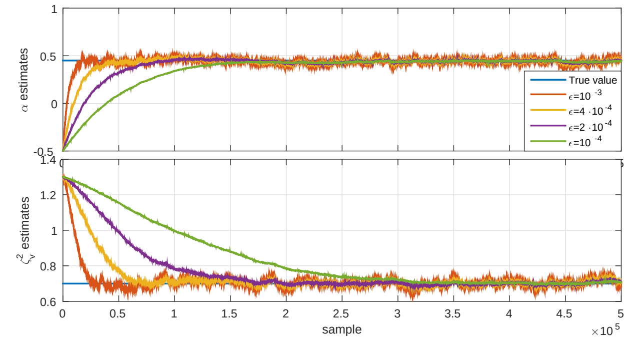

Figure 3. Numerical simulation of the example, for different values of ϵ.

straightforward, updating the noise variance ζ_v^2 requires a different definition of the prediction error. This can be done by using $\widehat{u}_{n-1,n}$ instead of $\widehat{u}_{n-1|n-1}$, which renders the entire procedure cumbersome.

Updating α is carried out by the recursive formulae

$$\alpha^{\mathrm{RPEM}}(n) = \alpha^{\mathrm{RPEM}}(n-1) + (n\cdot R(n))^{-1} \times\left(z_n\cdot\widehat{u}_{n-1|n-1} - \alpha^{\mathrm{RPEM}}(n-1)\cdot\widehat{u}_{n-1|n-1}^2\right) , \quad (45a)$$

$$R(n) = \frac{n-1}{n}\cdot R(n-1) + n^{-1}\cdot\widehat{u}_{n-1|n-1}^2 , \quad (45b)$$

It is evident that the second-order terms in (45) are obtained by the squared-value $\widehat{u}_{n-1|n-1}^2$, while in (43a) the covariance matrix given by the Kalman filter is used to estimate these second-order terms.

VI. Proof of the Main Theorem

Before providing the full proof of Theorem 9, we begin with a high level description. In the first part (Sec. VI-A), we express $\theta^\epsilon(t)$ as the sum of two terms, denoted by $\bar{G}^\epsilon(t)$ and $W^\epsilon(t)$, where the latter process is shown to be a Martingale. Although Sec. VI-A is similar to the proof of Theorem 8.2.1 in [9, p. 253], it is necessary for the tractability of the proof. Then, in Sec. VI-B, we use Martingale properties to show that $W^\epsilon(t)\to 0$ as $\epsilon\to 0$. Then, in Sec. VI-C we show that $\bar{G}^\epsilon(t)\to\bar{G}(t)$, where $\bar{G}(t)$ converges to a local minimum of the KLD defined in Asm. 2. The type of convergence of $W^\epsilon(t)$ and $\bar{G}^\epsilon(t)$ is discussed in detail within the proof.

A. Reformulating $\theta^\epsilon(t)$

Before continuing, the following definition and lemma are required.

Definition 11. *A continuous-time random process $U(t)$ is called local Lipschitz continuous (LLC) if for every T there exists a random variable u_T for which: $0 \le u_T < \infty$ w.p.1, and for every $0 \le t, s \le T$,*

$$|U(t+s) - U(t)| \le s\cdot u_T \;,\; \textit{w.p.1} . \quad (46)$$

Lemma 12. *Under Asm. 7, $\theta^\epsilon(t) \xrightarrow{D} \theta(t)$ as $\epsilon\to 0$, where $\theta(t)$ is an LLC random process.*

Proof. The proof is identical to Part 2 of the proof of Theorem 8.2.1 in [9, p. 253]. □

For pure analytical purposes, it would have been easier if we could use convergence with probability one, because we could then work directly with the sample paths of the processes; for example, the difference $\theta^\epsilon(t)-\theta(t)$. Since Theorem 9 ensures only a weak convergence, we resort to the Skorokhod representation (Theorem 6.7 in [88, p. 70]). By this representation, there exists a series of Skorokhod processes[7] $\widetilde{\theta}^\epsilon(t)$ and a process $\widetilde{\theta}(t)$, defined on a common probability space, for which

$$\Pr\{\widetilde{\theta}^\epsilon(t)\in A\}=\Pr\{\theta^\epsilon(t)\in A\},$$
$$\Pr\{\widetilde{\theta}(t)\in A\}=\Pr\{\theta(t)\in A\}\ , \tag{47}$$

for every group of paths A and

$$\widetilde{\theta}^\epsilon(t)\xrightarrow{\text{w.p.1}}\widetilde{\theta}(t). \tag{48}$$

Without loss of generality, we assume throughout the proof that $\theta^\epsilon(t)$ converges to $\theta(t)$ with probability one, where we formally refer to $\widetilde{\theta}^\epsilon(t)$ and $\widetilde{\theta}(t)$ respectively, and thus the conclusions hold only in distribution.

Now, define

$$\bar{g}_n(\theta)=E\left\{\nabla_\theta\log f\left(z_n|\mathcal{Z}_{n-1};\theta\right)\right\}\ , \tag{49}$$

and rewrite (24b) as

$$\theta^\epsilon(t)=\theta_0+\bar{G}^\epsilon(t)+W^\epsilon(t)\ , \tag{50a}$$
$$\bar{G}^\epsilon(t)\triangleq\epsilon\sum_{i=1}^{t/\epsilon}\bar{g}_i(\theta_i^\epsilon)\ , \tag{50b}$$
$$W^\epsilon(t)\triangleq\epsilon\sum_{i=1}^{t/\epsilon}\left[Y_i^\epsilon-\bar{g}_i(\theta_i^\epsilon)\right]\ . \tag{50c}$$

To find θ that minimizes the KLD between the parametric and true p.d.f.s, one can use its gradient, $\bar{g}_n(\theta)$, which is a deterministic function of θ. As in all SA algorithms, only random and noisy observations of $\bar{g}_n(\theta)$ are available, hence the approximated gradient steps Y_n^ϵ are used instead. In the following sections, we show that the limit $\bar{G}(t)\triangleq\lim_{\epsilon\to0}\bar{G}^\epsilon(t)$ exists, and that $W^\epsilon(t)\xrightarrow{\text{w.p.1}}0$ for all t, which together with Lemma 12 implies that $\theta(t)=\theta_0+\bar{G}(t)$; this is shown in Sec. VI-B. Then, in Sec. VI-C we investigate the relationships between $\bar{G}(t)$ and the KLD, $\bar{k}(\theta)$.

In many SA algorithms, the difference between the actual step (Y_n^ϵ in our case) and the desired step ($\bar{g}_n(\theta_n^\epsilon)$ in our case) is a Martingale difference, and the proof of convergence is henceforth straightforward. In our problem, the actual step is not only noisy, but is also biased by the approximation error in f_{n-1}^ϵ and the difference between Y_n^ϵ and $\bar{g}_n(\theta_n^\epsilon)$ is not a Martingale difference. Thus, we further decompose $W^\epsilon(t)$ into

$$W^\epsilon(t)=\widetilde{G}^\epsilon(t)+M^\epsilon(t)\ , \tag{51a}$$
$$\widetilde{G}^\epsilon(t)\triangleq\epsilon\sum_{i=1}^{t/\epsilon}\left[g_i^\epsilon(\theta_i^\epsilon)-\bar{g}_i(\theta_i^\epsilon)\right]\ , \tag{51b}$$
$$M^\epsilon(t)\triangleq\epsilon\sum_{i=1}^{t/\epsilon}\left[Y_i^\epsilon-g_i^\epsilon(\theta_i^\epsilon)\right]\ , \tag{51c}$$

where

$$g_n^\epsilon(\theta)=E\left\{Y_n^\epsilon(\theta)|\mathcal{Z}_{n-1}\right\}\ . \tag{52}$$

[7]Let $(\Omega,\mathcal{F},P)$ be a probability space. Then, a Skorokhod process is a function that maps every measurable set $A\in\mathcal{F}$ into the Borel σ-algebra w.r.t. the Skorokhod topology.

By definition, $M^\epsilon(t)$ is an $\mathcal{F}_{t/\epsilon}^\epsilon$-Martingale, where $\mathcal{F}_{t/\epsilon}^\epsilon$ is the filtration generated by $\left\{Y_j^\epsilon\right\}_{j=1}^{t/\epsilon}$. This fact is used to prove that $W^\epsilon(t)$ converges to zero, as shown in the next section.

B. The Limit of $W^\epsilon(t)$

Similar to Lemma 12 (and using the Skorokhod representation), it can be shown that since $\bar{g}_n(\theta)$ is uniformly integrable, the limit process $\bar{G}(t)$ exists; i.e., as ϵ approaches zero, $\bar{G}^\epsilon(t)$ converges w.p.1 to some LLC process $\bar{G}(t)$. This, together with Lemma 12, implies that there exists an LLC process $W(t)$ such that

$$\begin{aligned}W(t)&\triangleq\lim_{\epsilon\to0}W^\epsilon(t)\\&=\theta(t)-\theta_0-\bar{G}(t)\quad,\quad\text{w.p.1.}\end{aligned} \tag{53}$$

To show that $W(t)$ equals zero, we use the following lemma.

Lemma 13. *Consider $\widetilde{G}^\epsilon(t)$ and $W(t)$ defined in* (51b) *and* (53), *respectively. Then, if for every $t>0$ and $\tau>0$,*

$$E\left\{\widetilde{G}^\epsilon(t+\tau)-\widetilde{G}^\epsilon(t)\right\}\xrightarrow{\epsilon\to0}0, \tag{54}$$

then $W(t)=0$ for all t w.p.1.

Proof. See App. B. □

We now show that $\widetilde{G}^\epsilon(t)$ indeed satisfies (54) and therefore $W(t)=0$ for all t w.p.1. Recalling (51b), we define

$$\begin{aligned}\Delta\widetilde{G}^\epsilon(t,\tau)&\triangleq\widetilde{G}^\epsilon(t+\tau)-\widetilde{G}^\epsilon(t)\\&=\epsilon\sum_{i=t/\epsilon}^{(t+\tau)/\epsilon-1}\left[g_i^\epsilon(\theta_i^\epsilon)-\bar{g}_i(\theta_i^\epsilon)\right],\end{aligned} \tag{55}$$

and rewrite it as the following double sum

$$\Delta\widetilde{G}^\epsilon(t,\tau)=\epsilon\sum_{j\in J_{t,\tau,\epsilon}}\sum_{i\in I_{j,\epsilon}}\left[g_i^\epsilon(\theta_i^\epsilon)-\bar{g}_i(\theta_i^\epsilon)\right], \tag{56a}$$
$$I_{j,\epsilon}\triangleq\left\{jn_\epsilon,...,(j+1)n_\epsilon-1\right\}, \tag{56b}$$
$$J_{t,\tau,\epsilon}\triangleq\left\{t/(\epsilon n_\epsilon),...,(t+\tau)/(\epsilon n_\epsilon)-1\right\}, \tag{56c}$$

where $n_\epsilon>0$ (the number of elements of each set $I_{j,\epsilon}$) is chosen such that $n_\epsilon\xrightarrow{\epsilon\to0}\infty$ and $\epsilon n_\epsilon^2\xrightarrow{\epsilon\to0}0$, for instance $n_\epsilon=\frac{1}{\sqrt[p]{\epsilon}}$ with $p>2$. Note that the number of elements in each set increases, but nevertheless, the length of a set in time decreases. An illustration of the double-summation in (56) is depicted in Fig. 4. Later in the proof, it will be shown that as $\epsilon\to0$, the value of θ in the inner sum is approximately constant. This fact is then used to show that the actual gradient steps are (in average) equal to the optimal gradient steps.

Before we continue, two definitions are required. Given the conditional p.d.f.s $\phi_n(\cdot|\cdot,\cdot\ ;\theta)$ and $m_n(\cdot|\cdot,\cdot\ ;\theta)$, the recursion (20) can be viewed as a transformation of $f_{n-1}(x_{n-1}|\mathcal{Z}_{n-1};\theta)$ into $f_n(x_n|\mathcal{Z}_n;\theta)$, using θ and z_n. This transformation can be compactly defined as:

Definition 14. *Define the following transformation on the p.d.f. $f_{n-1}(\cdot|\mathcal{Z}_{n-1};\theta)$,*

$$\begin{aligned}&\mathcal{T}_{n,\theta}\left\{f_{n-1}\right\}(x_n)\triangleq\\&\frac{m_n(z_n|x_n,\mathcal{Z}_{n-N_m}^{n-1};\theta)f_n(x_n|\mathcal{Z}_{n-1};\theta)}{\int m_n(z_n|x_n,\mathcal{Z}_{n-N_m}^{n-1};\theta)f_n(x_n|\mathcal{Z}_{n-1};\theta)dx_n}\ ,\end{aligned} \tag{57a}$$

The final version of record is available at http://dx.doi.org/10.1109/TSP.2018.2889945

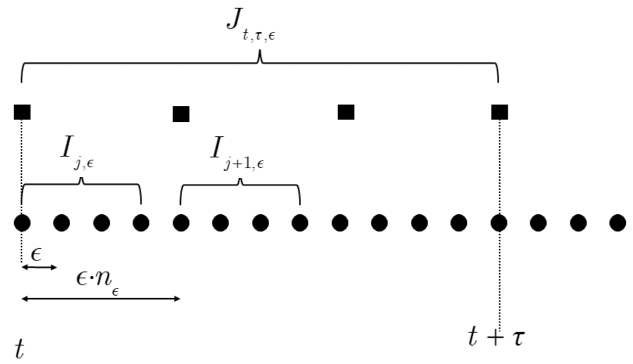

Figure 4. Double summation illustration. Note that the inner summation involves $i \in I_{j,\epsilon}$ and the outer summation involves $j \in J_{t,\tau,\epsilon}$.

where

$$f_n(x_n|\mathcal{Z}_{n-1};\theta) = \int \phi_n(x_n|x_{n-1}, \mathcal{Z}_{n-N_\phi}^{n-1};\theta) f_{n-1}(x_{n-1}|\mathcal{Z}_{n-1};\theta) dx_{n-1} \ . \tag{57b}$$

In the following, we define a fixed-θ recursion, which is similar to the RML but without updating the parameters; i.e., the p.d.f.s are updated according to (21a) – (21d), but the parameter update (21e) – (21f) is not executed. This recursion is used to approximate the RML for very small steps; i.e., when $\epsilon \to 0$.

Definition 15. ***(Fixed-θ recursion)*** *Let $f_{q,q}(x_q|\mathcal{Z}_q;\theta)$ be the p.d.f. of x_q given $\mathcal{Z}_q$. For $n > q$, denote by $f_{n,q}(x_n|\mathcal{Z}_n;\theta)$, the p.d.f. of x_n given $\mathcal{Z}_n$ that is obtained by $n-q$ subsequent applications of the transformation in Def. 14 for a fixed θ. In more explicit terms,*

$$\begin{aligned} f_{q+1,q}(x_{q+1}|\mathcal{Z}_{q+1};\theta) &= \mathcal{T}_{q+1,\theta}\left\{f_{q,q}(x_q|\mathcal{Z}_q;\theta)\right\}, \\ f_{q+2,q}(x_{q+2}|\mathcal{Z}_{q+2};\theta) &= \mathcal{T}_{q+2,\theta}\left\{f_{q+1,q}(x_{q+1}|\mathcal{Z}_{q+1};\theta)\right\}, \\ &\vdots \\ f_{n,q}(x_n|\mathcal{Z}_n;\theta) &= \mathcal{T}_{n,\theta}\left\{f_{n-1,q}(x_{n-1}|\mathcal{Z}_{n-1};\theta)\right\} \ . \end{aligned} \tag{58a}$$

The following intermediate p.d.f.s are also used,

$$f_{n,q}(x_n|\mathcal{Z}_{n-1};\theta) = \int \phi_n(x_n|x_{n-1}, \mathcal{Z}_{n-N_\phi}^{n-1};\theta) \times f_{n-1,q}(x_{n-1}|\mathcal{Z}_{n-1};\theta) dx_{n-1}, \tag{58b}$$

$$f_{n,q}(z_n, x_n|\mathcal{Z}_{n-1};\theta) = m_n(z_n|x_n, \mathcal{Z}_{n-N_m}^{n-1};\theta) \times f_{n,q}(x_n|\mathcal{Z}_{n-1};\theta), \tag{58c}$$

and similar to (52)*, denote*

$$g_{n,q}(\theta) \triangleq E\left\{Y_{n,q}(\theta)|\mathcal{Z}_{n-1}\right\}, \tag{59a}$$

where

$$Y_{n,q}(\theta) = \int \nabla_\theta \log f_{n,q}(z_n, x_n|\mathcal{Z}_{n-1};\theta) \times f_{n,q}(x_n|\mathcal{Z}_n;\theta)\ dx_n \ . \tag{59b}$$

To use Lemma 13 we need to show that $E\left\{\Delta\widetilde{G}^\epsilon(t,\tau)\right\}$ converges to zero, which is done in two steps. First, we define

$$\Delta\widehat{G}^\epsilon(t,\tau) \triangleq \epsilon \sum_{j\in J_{t,\tau,\epsilon}} \sum_{i\in I_{j,\epsilon}} \left[g_{i,jn_\epsilon}(\theta^\epsilon_{jn_\epsilon}) - \bar{g}_i(\theta^\epsilon_{jn_\epsilon})\right] \ , \tag{60}$$

and show that $|\Delta\widehat{G}^\epsilon(t,\tau) - \Delta\widetilde{G}^\epsilon(t,\tau)| \xrightarrow{\text{w.p.1}} 0$; this is done in Props. 16 and 17. Then, in Lemma 18, we show that $E\{\Delta\widetilde{G}^\epsilon(t,\tau)\} \xrightarrow{\epsilon\to 0} 0$.

Lemma 16. *Let*

$$\Delta\widehat{G}^\epsilon(t,\tau) = \widehat{G}^\epsilon_2(t,\tau) - \widehat{G}^\epsilon_1(t,\tau), \tag{61}$$

$$\Delta\widetilde{G}^\epsilon(t,\tau) = \widetilde{G}^\epsilon_2(t,\tau) - \widetilde{G}^\epsilon_1(t,\tau), \tag{62}$$

where

$$\widehat{G}^\epsilon_1(t,\tau) \triangleq \epsilon \sum_{j\in J_{t,\tau,\epsilon}} \sum_{i\in I_{j,\epsilon}} \bar{g}_i(\theta^\epsilon_{jn_\epsilon}), \tag{63a}$$

$$\widetilde{G}^\epsilon_1(t,\tau) \triangleq \epsilon \sum_{j\in J_{t,\tau,\epsilon}} \sum_{i\in I_{j,\epsilon}} \bar{g}_i(\theta^\epsilon_i), \tag{63b}$$

$$\widehat{G}^\epsilon_2(t,\tau) \triangleq \epsilon \sum_{j\in J_{t,\tau,\epsilon}} \sum_{i\in I_{j,\epsilon}} g_{i,jn_\epsilon}(\theta^\epsilon_{jn_\epsilon}), \tag{63c}$$

$$\widetilde{G}^\epsilon_2(t,\tau) \triangleq \epsilon \sum_{j\in J_{t,\tau,\epsilon}} \sum_{i\in I_{j,\epsilon}} g^\epsilon_i(\theta^\epsilon_i). \tag{63d}$$

For every $t, \tau < \infty$, $|\widehat{G}^\epsilon_1(t,\tau) - \widetilde{G}^\epsilon_1(t,\tau)| \xrightarrow{w.p.1} 0$.

Proof. For every ϵ, t, and τ,

$$\begin{aligned} \left|\widehat{G}^\epsilon_1(t,\tau) - \widetilde{G}^\epsilon_1(t,\tau)\right| &\le \epsilon \sum_{j\in J_{t,\tau,\epsilon}} \sum_{i\in I_{j,\epsilon}} \left|\bar{g}_i(\theta^\epsilon_{jn_\epsilon}) - \bar{g}_i(\theta^\epsilon_i)\right| \\ &\le \tau \cdot \sup_{\substack{j\in J_{t,\tau,\epsilon} \\ i\in I_{j,\epsilon}}} \left|\bar{g}_i(\theta^\epsilon_{jn_\epsilon}) - \bar{g}_i(\theta^\epsilon_i)\right| \ , \end{aligned} \tag{64a}$$

where in the latter inequality, we used the total number of elements in the double summation, τ/ϵ. By (24a), for every t in the range $t \in [j\epsilon n_\epsilon, j\epsilon n_\epsilon + \epsilon)$, the continuous process is equal to $\theta^\epsilon(t) = \theta^\epsilon_{jn_\epsilon}$; thus the discrete range $i \in I_{j,\epsilon}$ is equivalent to the continuum $t' \in [j\epsilon n_\epsilon, j\epsilon n_\epsilon + \epsilon n_\epsilon)$. This implies that the equivalent time difference between $\theta^\epsilon_{jn_\epsilon}$ and θ^ϵ_i in (64a) is bounded by ϵn_ϵ, hence

$$\sup_{\substack{j\in J_{t,\tau,\epsilon} \\ i\in I_{j,\epsilon}}} \left|\bar{g}_i(\theta^\epsilon_{jn_\epsilon}) - \bar{g}_i(\theta^\epsilon_i)\right| \le \sup_{\substack{i\in\{t/\epsilon,\dots,(t+\tau)/\epsilon\} \\ t'\in[t,t+\tau]\ ,\ t''\in[t',t'+\epsilon n_\epsilon)}} \left|\bar{g}_i(\theta^\epsilon(t'')) - \bar{g}_i(\theta^\epsilon(t'))\right| \ . \tag{64b}$$

Next, we note that

$$|\theta^\epsilon(t'') - \theta^\epsilon(t')| \le \underbrace{|\theta^\epsilon(t'') - \theta(t'')|}_{\triangleq|\Delta\theta^\epsilon(t'')|} + \underbrace{|\theta^\epsilon(t') - \theta(t')|}_{\triangleq|\Delta\theta^\epsilon(t')|} + |\theta(t'') - \theta(t')| , \tag{64c}$$

and using the Skorokhod representation of Lemma 12, $|\Delta\theta^\epsilon(t')|$ and $|\Delta\theta^\epsilon(t'')|$ converges to zero w.p.1. Now, because $\theta(t)$ is LLC, there exists $u(t+\tau) < \infty$ such that for every $\epsilon n_\epsilon > 0$

$$\sup_{\substack{t'\in[t,t+\tau] \\ t''\in[t',t'+\epsilon n_\epsilon)}} |\theta(t'') - \theta(t')| \le |\epsilon n_\epsilon \cdot u(t+\tau)| , \tag{64d}$$

and by the choice of n_ϵ, the right-hand side of (64d) diminishes as ϵ approaches zero for every t and τ, w.p.1. Under Assumptions 4 and 8, $\bar{g}_n(\theta)$ is continuous w.r.t. θ, uniformly in n; therefore by the continuous mapping theorem (Theorem 2.3 in [89, p. 7]), for every t and τ

$$\sup_{\substack{i\in[t/\epsilon,(t+\tau)/\epsilon] \\ t'\in[t,t+\tau]\ ,\ t''\in[t',t'+\epsilon n_\epsilon]}} |\bar{g}_i(\theta^\epsilon(t'')) - \bar{g}_i(\theta^\epsilon(t'))| \xrightarrow[\epsilon\to 0]{w.p.1} 0 \ , \tag{64e}$$

establishing Lemma 16. □

Lemma 17. *For every $t,\tau \le \infty$, $|\widehat{G}_2^\epsilon(t,\tau) - \widetilde{G}_2^\epsilon(t,\tau)| \xrightarrow{w.p.1} 0$.*

Proof. In Lemma 16, we investigated $\left|\widehat{G}_1^\epsilon(t,\tau) - \widetilde{G}_1^\epsilon(t,\tau)\right|$ by analyzing $\left|\bar{g}_i(\theta^\epsilon_{jn_\epsilon}) - \bar{g}_i(\theta^\epsilon_i)\right|$ (see (64a)). Similarly, for Lemma 17 it is sufficient to show that

$$\sup_{\substack{j\in J_{t,\tau,\epsilon}\\ i\in I_{j,\epsilon}}} \left|g_i^\epsilon(\theta_i^\epsilon) - g_{i,jn_\epsilon}(\theta^\epsilon_{jn_\epsilon})\right| \xrightarrow{\text{w.p.1}} 0 \ . \tag{65a}$$

Given Lemma 12 and the uniform continuity of g_i^ϵ, the latter is equivalent to

$$\sup_{\substack{j\in J_{t,\tau,\epsilon}\\ i\in I_{j,\epsilon}}} \left|g_i^\epsilon(\theta^\epsilon_{jn_\epsilon}) - g_{i,jn_\epsilon}(\theta^\epsilon_{jn_\epsilon})\right| \xrightarrow{\text{w.p.1}} 0 \ . \tag{65b}$$

To prove (65b), we examine the expression in the argument of the absolute value,

$$\begin{aligned} &g_i^\epsilon(\theta^\epsilon_{jn_\epsilon}) - g_{i,jn_\epsilon}(\theta^\epsilon_{jn_\epsilon}) = \\ &\quad E\left\{Y_i^\epsilon(\theta^\epsilon_{jn_\epsilon}) - Y_{i,jn_\epsilon}(\theta^\epsilon_{jn_\epsilon})\middle|\mathcal{Z}_{i-1}\right\} = \\ &\quad E\left\{\int \left(\nabla_\theta \log f_i^\epsilon(z_i,x_i|\mathcal{Z}_{i-1};\theta^\epsilon_{jn_\epsilon})\right)\times \right. \\ &\qquad\qquad \left. \left[\mathcal{T}^i_{jn_\epsilon}[\theta^\epsilon_{jn_\epsilon+1},...,\theta^\epsilon_i] - \mathcal{T}^i_{jn_\epsilon}[\theta^\epsilon_{jn_\epsilon},...,\theta^\epsilon_{jn_\epsilon}]\right] dx_i \middle|\mathcal{Z}_{i-1}\right\} , \end{aligned} \tag{65c}$$

where

$$\begin{aligned} &\mathcal{T}^i_{jn_\epsilon}[\theta^\epsilon_{jn_\epsilon},...,\theta^\epsilon_i] \\ &\quad \triangleq \mathcal{T}_{i,\theta_i^\epsilon}\left\{\mathcal{T}_{i-1,\theta^\epsilon_{i-1}}\left\{\cdots \mathcal{T}_{jn_\epsilon+1,\theta^\epsilon_{jn_\epsilon+1}}\left\{f^\epsilon_{jn_\epsilon}\right\}\right\}\right\} \ . \end{aligned} \tag{65d}$$

By Asm. 7, (65c) is established by showing

$$\left|\mathcal{T}^i_{jn_\epsilon}[\theta^\epsilon_{jn_\epsilon+1},...,\theta^\epsilon_i] - \mathcal{T}^i_{jn_\epsilon}[\theta^\epsilon_{jn_\epsilon},...,\theta^\epsilon_{jn_\epsilon}]\right| \xrightarrow{\text{w.p.1}} 0 \ . \tag{65e}$$

To do so, we rewrite the latter as

$$\begin{aligned} &\left|\mathcal{T}^i_{jn_\epsilon}[\theta^\epsilon_{jn_\epsilon+1},...,\theta^\epsilon_i] - \mathcal{T}^i_{jn_\epsilon}[\theta^\epsilon_{jn_\epsilon},...,\theta^\epsilon_{jn_\epsilon}]\right| \\ &\le \left|\mathcal{T}^i_{jn_\epsilon}[\theta^\epsilon_{jn_\epsilon+1},...,\theta^\epsilon_i] - \mathcal{T}^i_{jn_\epsilon}[\theta(j\epsilon n_\epsilon+\epsilon),...,\theta(i\epsilon)]\right| \\ &+ \left|\mathcal{T}^i_{jn_\epsilon}[\theta(j\epsilon n_\epsilon+\epsilon),...,\theta(i\epsilon)] - \mathcal{T}^i_{jn_\epsilon}[\theta(j\epsilon n_\epsilon),...,\theta(j\epsilon n_\epsilon)]\right| \ , \\ &+ \left|\mathcal{T}^i_{jn_\epsilon}[\theta(j\epsilon n_\epsilon),...,\theta(j\epsilon n_\epsilon)] - \mathcal{T}^i_{jn_\epsilon}[\theta^\epsilon_{jn_\epsilon},...,\theta^\epsilon_{jn_\epsilon}]\right| \end{aligned} \tag{65f}$$

and then show that each term on the right-hand side converges to zero.

The first term on the right-hand side of (65f) satisfies

$$\begin{aligned} &\left|\mathcal{T}^i_{jn_\epsilon}[\theta^\epsilon_{jn_\epsilon+1},...,\theta^\epsilon_i] - \mathcal{T}^i_{jn_\epsilon}[\theta(j\epsilon n_\epsilon+\epsilon),...,\theta(i\epsilon-\epsilon),\theta(i\epsilon)]\right| \\ &\le \left|\mathcal{T}^i_{jn_\epsilon}[\theta^\epsilon_{jn_\epsilon+1},...,\theta^\epsilon_i] - \mathcal{T}^i_{jn_\epsilon}[\theta(j\epsilon n_\epsilon+\epsilon),\theta^\epsilon_{jn_\epsilon+2},...,\theta^\epsilon_i]\right| \\ &+ \left|\mathcal{T}^i_{jn_\epsilon}[\theta(j\epsilon n_\epsilon+\epsilon),\theta^\epsilon_{jn_\epsilon+2},...,\theta^\epsilon_i]\right. \\ &\qquad \left. -\mathcal{T}^i_{jn_\epsilon}[\theta(j\epsilon n_\epsilon+\epsilon),\theta(j\epsilon n_\epsilon+2\epsilon),...,\theta^\epsilon_i]\right| \\ &\quad \vdots \\ &+ \left|\mathcal{T}^i_{jn_\epsilon}[\theta(j\epsilon n_\epsilon+\epsilon),...,\theta(i\epsilon-\epsilon),\theta^\epsilon_i]\right. \\ &\qquad \left. -\mathcal{T}^i_{jn_\epsilon}[\theta(j\epsilon n_\epsilon+\epsilon),...,\theta(i\epsilon-\epsilon),\theta(i\epsilon)]\right| \ , \end{aligned} \tag{65g}$$

or in words, it can be written as a sum of differences, where in each term, only a single argument of $\mathcal{T}$ is different. Consider one of the terms on the right-hand side of (65g), for example

$$\left|\mathcal{T}^i_{jn_\epsilon}[\theta^\epsilon_{jn_\epsilon+1},...,\theta^\epsilon_i] - \mathcal{T}^i_{jn_\epsilon}[\theta(j\epsilon n_\epsilon+\epsilon),\theta^\epsilon_{jn_\epsilon+2},...,\theta^\epsilon_i]\right| \ . \tag{65h}$$

By Asm. 3, $\mathcal{T}^i_{jn_\epsilon}$ is continuous w.r.t. its first argument, and hence from Lemma 12 this term converges to zero as $\epsilon\to 0$, w.p.1., i.e., there exists a series $\delta_{\epsilon,1} \xrightarrow{\epsilon\to 0} 0$ such that for every ϵ,

$$\left|\mathcal{T}^i_{jn_\epsilon}[\theta^\epsilon_{jn_\epsilon+1},...,\theta^\epsilon_i] - \mathcal{T}^i_{jn_\epsilon}[\theta(j\epsilon n_\epsilon+\epsilon),\theta^\epsilon_{jn_\epsilon+2},...,\theta^\epsilon_i]\right| \le \delta_{\epsilon,1} \ . \tag{65i}$$

Since $\mathcal{T}^i_{jn_\epsilon}$ is uniformly continuous w.r.t. each of its arguments, a similar inequality holds for each of the terms on the right-hand side of (65g), i.e., there exists another series $\delta_\epsilon \xrightarrow{\epsilon\to 0} 0$ such that for every ϵ,

$$\begin{aligned} &\left|\mathcal{T}^i_{jn_\epsilon}[\theta^\epsilon_{jn_\epsilon+1},...,\theta^\epsilon_i]\right. \\ &\qquad \left. -\mathcal{T}^i_{jn_\epsilon}[\theta(j\epsilon n_\epsilon+\epsilon),...,\theta(i\epsilon)]\right| \le n_\epsilon\delta_\epsilon \ . \end{aligned} \tag{65j}$$

What remains is to choose a frame size that satisfies $n_\epsilon \xrightarrow{\epsilon\to 0} \infty$ slowly enough such that $n_\epsilon\delta_\epsilon \xrightarrow{\epsilon\to 0} 0$. The last term on the right-hand side of (65f) converges to zero using the same arguments.

The third term on the right-hand side of (65f) satisfies

$$\begin{aligned} &\left|\mathcal{T}^i_{jn_\epsilon}[\theta(j\epsilon n_\epsilon+\epsilon),...,\theta(i\epsilon)] - \mathcal{T}^i_{jn_\epsilon}[\theta(j\epsilon n_\epsilon),...,\theta(j\epsilon n_\epsilon)]\right| \\ &\le \left|\mathcal{T}^i_{jn_\epsilon}[\theta(j\epsilon n_\epsilon+\epsilon),\theta(j\epsilon n_\epsilon+2\epsilon),...,\theta(i\epsilon)]\right. \\ &\qquad \left. -\mathcal{T}^i_{jn_\epsilon}[\theta(j\epsilon n_\epsilon),\theta(j\epsilon n_\epsilon+2\epsilon),...,\theta(i\epsilon)]\right| \\ &+ \left|\mathcal{T}^i_{jn_\epsilon}[\theta(j\epsilon n_\epsilon),\theta(j\epsilon n_\epsilon+2\epsilon),...,\theta(i\epsilon)]\right. \\ &\qquad \left. -\mathcal{T}^i_{jn_\epsilon}[\theta(j\epsilon n_\epsilon),\theta(j\epsilon n_\epsilon),\theta(j\epsilon n_\epsilon+3\epsilon),...,\theta(i\epsilon)]\right| \\ &\quad \vdots \\ &+ \left|\mathcal{T}^i_{jn_\epsilon}[\theta(j\epsilon n_\epsilon),...,\theta(j\epsilon n_\epsilon),\theta(i\epsilon)]\right. \\ &\qquad \left. -\mathcal{T}^i_{jn_\epsilon}[\theta(j\epsilon n_\epsilon),...,\theta(j\epsilon n_\epsilon)]\right| \ . \end{aligned} \tag{65k}$$

By Lemma 12, and the continuity of $\mathcal{T}$, each of the terms on the right-hand side of (65k) is bounded by $D\epsilon n_\epsilon$, where D is some random variable satisfying $D<\infty$ w.p.1. Therefore, the combination of all the right-hand side terms in (65k) implies

$$\begin{aligned} &\left|\mathcal{T}^i_{jn_\epsilon}[\theta(j\epsilon n_\epsilon+\epsilon),...,\theta(i\epsilon)]\right. \\ &\qquad \left. -\mathcal{T}^i_{jn_\epsilon}[\theta(j\epsilon n_\epsilon),...,\theta(j\epsilon n_\epsilon)]\right| \le D\epsilon n_\epsilon^2 \ . \end{aligned} \tag{65l}$$

As before, the length of frame can be chosen such that $n_\epsilon \xrightarrow{\epsilon\to 0} \infty$ and yet $\epsilon n_\epsilon^2 \xrightarrow{\epsilon\to 0} 0$. □

From Lemmas 16 and 17 it follows that $|\Delta\widetilde{G}^\epsilon(t,\tau) - \Delta\widehat{G}^\epsilon(t,\tau)| \xrightarrow{D} 0$. It remains to show that (54) is satisfied, for which we introduce the following lemma:

Lemma 18. *Consider $\Delta\widetilde{G}^\epsilon(t,\tau)$ defined in (55), then for every $t,\tau<\infty$, $E\{\Delta\widetilde{G}^\epsilon(t,\tau)\} \xrightarrow{\epsilon\to 0} 0$.*

Proof. From Lemmas 16 and 17,

$$E\left\{\Delta\widetilde{G}^\epsilon(t,\tau) - \Delta\widehat{G}^\epsilon(t,\tau)\right\} \xrightarrow{\epsilon\to 0} 0 \ . \tag{66a}$$

Thus, it is sufficient to show that

$$E\left\{\Delta\widehat{G}^\epsilon(t,\tau)\right\} \xrightarrow{\epsilon\to 0} 0 \ . \tag{66b}$$

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For the following analysis, recall equations (49), (59), and (60). Similar to (17), $Y_{n,q}(\theta)$ can be written as

$$Y_{n,q}(\theta) = \nabla_\theta \log f_{n,q}(z_n|\mathcal{Z}_{n-1}) \ , \qquad (66c)$$

and by the law of total expectation it follows that

$$E\left\{\bar{g}_n(\theta) - g_{n,q}(\theta, f_{q,q})\right\} = \nabla_\theta E\left\{\log \frac{f_n(z_n|\mathcal{Z}_{n-1};\theta)}{f_{n,q}(z_n|\mathcal{Z}_{n-1};\theta)}\right\} \ , \qquad (66d)$$

where we also used Asm. 8 to exchange the order of expectation and gradient. Under Asm. 5, the right-hand side of (66d) converges to zero as $n \to \infty$, implying

$$E\left\{\Delta\widehat{G}^\epsilon(t,\tau)\right\} = \epsilon \sum_{i=1}^{t/\epsilon} E\left\{g_{i,jn_\epsilon}(\theta^\epsilon_{jn_\epsilon}) - \bar{g}_i(\theta^\epsilon_{jn_\epsilon})\right\} \xrightarrow{\epsilon\to 0} 0 \ .$$

□

C. Finalé

By Lemmas 18 and 13 it follows that w.p.1, $W(t) = 0$, $\forall t$, thus from (53) it follows that

$$\theta(t) = \theta_0 + \bar{G}(t) \ . \qquad (67)$$

We now show that w.p.1, $\bar{G}(t)$ is a solution of an ODE, and that the solution of this ODE is a stationary point of $\bar{k}_n(\theta)$. To this end, we use an alternative expression for $\bar{G}(t)$. By Asm. 8, $\bar{g}_n(\theta) = -\nabla_\theta \bar{k}_n(\theta)$, and by further employing Asm. 2, it follows that for every θ, $\bar{g}_n(\theta) \xrightarrow{n\to\infty} \bar{g}(\theta) = -\nabla_\theta \bar{k}(\theta)$. Thus for every $t > 0$,

$$\bar{G}(t) = \lim_{\epsilon\to 0} \bar{G}^\epsilon(t) = \lim_{\epsilon\to 0} \epsilon \sum_{i=1}^{t/\epsilon - 1} \bar{g}(\theta^\epsilon_i) \ , \quad \text{w.p.1.} \ . \qquad (68)$$

By Lemma 12, (68), the continuity of $\bar{g}(\theta)$, and by invoking Lebesgue's criterion for the Riemann integrability,

$$\lim_{\epsilon\to 0} \bar{G}^\epsilon(t) = \int_0^t \bar{g}[\theta(s)]ds \ . \qquad (69)$$

Substituting $\bar{G}(t)$ into (67) and taking the derivative w.r.t. t, we obtain $\frac{\partial}{\partial t}\theta(t) = \bar{g}[\theta(t)] = -\nabla_\theta \bar{k}[\theta(t)]$. Finally, the derivative of $\bar{k}[\theta(t)]$ w.r.t. t is

$$\begin{aligned}\tfrac{\partial}{\partial t}\bar{k}[\theta(t)] &= \nabla_\theta \bar{k}^H[\theta(t)] \cdot \tfrac{\partial}{\partial t}\theta(t) \\ &= -\nabla_\theta \bar{k}^H[\theta(t)] \cdot \nabla_\theta \bar{k}[\theta(t)] < 0 \ . \end{aligned} \qquad (70)$$

Since $\bar{k}[\theta(t)]$ is a non-negative, monotonically decreasing function of t, it follows that $\theta(t)$ converges to a stationary set of $\bar{k}(\theta)$.

VII. Conclusion

A gradient-based RML algorithm was formulated for a general statistical model, which includes unknown parameters, continuous hidden data, and time dependent observations. The algorithm calculates the propagation of the p.d.f.s by analytically computing the integrals w.r.t. the state variables. In the discrete case, this solution is similar to other methods for HMM, which have been analytically investigated in earlier works. In the Gaussian case, numerous problems of this kind have been addressed in the literature, but without a rigorous formulation and analysis. We provided an analytical filtering formulae and proved the convergence of the RML in cases of practical importance.

The purpose of the proposed algorithm is to minimize the KLD between the conditional p.d.f. $h_n(z_n|\mathcal{Z}_n)$ and the respective modelled p.d.f. $f_n(z_n|\mathcal{Z};\theta)$, which is asymptotically equivalent to the maximization of the likelihood. A full convergence proof was given, based on the ODE approach, where it was shown that the series of estimates converges weakly to a stationary point of the KLD, even for a miss-specified statistical model. The technical assumptions required for convergence deal with two natural aspects of the problem: (a) the stationarity of the hidden and observed stochastic processes, and (b) the feasibility of solving the involved integrals and the calculation of the gradient. We also presented examples, to demonstrate the applicability of the recursive algorithm, and the technical assumptions were shown to hold. An extension of the algorithm to a constrained problem was presented as well. By targeting the rate and accuracy of the method, future work might be devoted to a Newton-based recursion instead of the gradient-based recursion, or updating the parameters with a time-averaged version of the gradient instead of updating the parameter at every time sample. In light of recent developments in stochastic optimization [20], we consider the investigation of the asymptotic properties of the RML as a very important subject for future research.

Appendix A
Extension of Fisher's identity

In the following we prove (17), which is an extension of the *Fisher identity* (9) (Sec. 15 in [53]), by using Asm. 8.

Proof. Using Bayes' theorem,

$$\begin{aligned} &E_\theta\left\{\nabla_\theta \log f\left(z_n, x_n|\mathcal{Z}_{n-1};\theta\right)|\mathcal{Z}_n\right\} \\ &= \int \nabla_\theta \log f\left(z_n|\mathcal{Z}_{n-1};\theta\right) \cdot f\left(x_n|\mathcal{Z}_n;\theta\right) dx_n \\ &\quad + \int \nabla_\theta \log f\left(x_n|\mathcal{Z}_n;\theta\right) \cdot f\left(x_n|\mathcal{Z}_n;\theta\right) dx_n \ . \end{aligned} \qquad (71)$$

The latter term equals $\int \nabla_\theta f\left(x_n|\mathcal{Z}_n;\theta\right) dx_n$, which in turn equals zero by using Asm. 8 to replace the gradient and integral order. Finally, (71) is

$$\begin{aligned} &\nabla_\theta \log f\left(z_n|\mathcal{Z}_{n-1};\theta\right) \int f\left(x_n|\mathcal{Z}_n;\theta\right) dx_n \\ &\qquad = \nabla_\theta \log f\left(z_n|\mathcal{Z}_{n-1};\theta\right) \ . \end{aligned} \qquad (72)$$

Appendix B
Proof of Lemma 13

This appendix is based on Theorems 4.1.1 and 7.4.1 from [9], and the proof of Theorem 8.2.1 therein. For the sake of clarity, these components are mentioned in order.

Proof. In the following, there are two steps: first, it is shown that $W(t)$ is a Martingale, and second, that $W(t) = 0$. To show that $W(t)$ is a Martingale, the following theorem is used.

Theorem 19. *[9, p. 234] Let $V(\cdot)$ be an r-dimensional random process, and $\mathcal{F}^V_t$ be the σ-algebra generated by $\{V(s) : s \le t\}$. For every integer p and time t, let $T_p(t) \triangleq \{t_1, t_2, ..., t_p\}$, where $t_i \le t$, $\forall i = 1, ..., p$. Further let $\xi : \mathcal{R}^{r\times p} \to \mathcal{R}$ be a continuous real-valued function.*

Denote by $D[0,\infty)$ the Skorokhod space; i.e., the space of real-valued functions defined on the interval $[0,\infty)$ that are right-continuous and have left-hand limits, with the Skorokhod topology used, and $D^r[0,\infty)$, its r-fold product. Now, let $U(t)$

be a random process with paths in $D^r[0,\infty)$, measurable on $\mathcal{F}_t^V$, with $E\,|U(t)| < \infty$ for every t.

Suppose that for every $t \geq 0$ and $\tau \geq 0$, each integer p, and every ξ, the following equation holds

$$E\left(\xi\left[V(t_i) : t_i \in T_p(t)\right] \cdot \left[U(t+\tau) - U(t)\right]\right) = 0 \; . \tag{73}$$

***Then**, $U(t)$ is an $\mathcal{F}_t^V$-Martingale.*

Considering (51a), for every bounded and continuous real-valued function ξ,

$$\begin{aligned} 0 = &\overbrace{E\left\{\xi\left[\theta^\epsilon(s_i), s_i \in T_p(t)\right] \cdot \left[W^\epsilon(t+\tau) - W^\epsilon(t)\right]\right\}}^{\triangleq \omega^\epsilon_{t,\tau}} \\ &- \overbrace{E\left\{\xi\left[\theta^\epsilon(s_i), s_i \in T_p(t)\right] \cdot \left[M^\epsilon(t+\tau) - M^\epsilon(t)\right]\right\}}^{\triangleq \mu^\epsilon_{t,\tau}} \\ &- \overbrace{E\left\{\xi\left[\theta^\epsilon(s_i), s_i \in T_p(t)\right] \cdot \left[\widetilde{G}^\epsilon(t+\tau) - \widetilde{G}^\epsilon(t)\right]\right\}}^{\triangleq \gamma^\epsilon_{t,\tau}} . \end{aligned} \tag{74}$$

We now take $\epsilon \to 0$ on both sides of (74) and prove that $\mu^\epsilon_{t,\tau}$ and $\gamma^\epsilon_{t,\tau}$ converge to zero, implying that $W(t)$ is a Martingale by Theorem 19. We start by proving $\mu^\epsilon_{t,\tau} \to 0$; using the law of total expectation [90],

$$\begin{aligned} &E\left\{\xi\left[\theta^\epsilon(s_i), s_i \in T_p(t)\right] \cdot \left[M^\epsilon(t+\tau) - M^\epsilon(t)\right]\right\} \\ &= E\left\{\xi\left[\theta^\epsilon(s_i), s_i \in T_p(t)\right]\right. \\ &\quad \left. \times E\left[M^\epsilon(t+\tau) - M^\epsilon(t) \middle| \theta^\epsilon_j, Y^\epsilon_j, j \leq t/\epsilon\right]\right\} \;, \end{aligned}$$

where the right-hand side is zero because $M^\epsilon(t)$ is an $\mathcal{F}^\epsilon_{t/\epsilon}$-Martingale. Showing that $\gamma^\epsilon_{t,\tau} \to 0$, we use the fact that ξ is bounded by definition; thus there exists C_ξ such that

$$\begin{aligned} &\left| E\left\{\xi\left[\theta^\epsilon(s_i), s_i \in T_p(t)\right] \cdot \left[\widetilde{G}^\epsilon(t+\tau) - \widetilde{G}^\epsilon(t)\right]\right\}\right| \leq \\ &\qquad C_\xi \cdot \left| E\left\{\widetilde{G}^\epsilon(t+\tau) - \widetilde{G}^\epsilon(t)\right\}\right| \; . \end{aligned} \tag{75}$$

Now recall that in Lemma 13 it is assumed that $E\left\{\widetilde{G}^\epsilon(t+\tau) - \widetilde{G}^\epsilon(t)\right\} \xrightarrow{\epsilon\to 0} 0$, hence $\gamma^\epsilon_{t,\tau} \xrightarrow{\epsilon\to 0} 0$, implying $\omega^\epsilon_{t,\tau} \xrightarrow{\epsilon\to 0} 0$. Now that we have shown that $W(t)$ is an $\mathcal{F}^\epsilon_{t/\epsilon}$-Martingale, we show that it is zero for every t by using the following theorem,

Theorem 20. *[9, p. 98] A continuous-time Martingale whose sample paths are LLC with probability one is constant with probability one.*

It was shown in (53) that $W(t)$ is LLC, and by Theorem 20 we obtain that $W(t)$ is constant w.p.1. By definition, $W(0) = 0$ and hence $W(t) = 0$ for every t.

Appendix C
Proof of Corollary 10

The proof of Corollary 10 is almost identical to the proof of Theorem 9, and we only note the differences. Start with the definition $R^\epsilon_n \triangleq \Pi(Y^\epsilon_n) - Y^\epsilon_n$, which is simply the difference between the constrained and unconstrained steps, and further define its interpolated version $R^\epsilon(t) = \epsilon \sum_{i=1}^{t/\epsilon} R^\epsilon_n$. By [9], Lemma 12 also holds if $\theta^\epsilon(t) + R^\epsilon(t)$ is used instead of $\theta^\epsilon(t)$, where the limit process is now $\theta(t) + R(t)$, with $R(t) = \lim_{\epsilon\to 0} R^\epsilon(t)$. Therefore, instead of (67), we get $\theta(t) + R(t) = \theta_0 + \bar{G}(t)$, and instead of (70) we have that

$$\frac{\partial}{\partial t}\bar{k}[\theta(t) + R(t)] = -\nabla_\theta \bar{k}^H[\theta(t)] \cdot \nabla_\theta \bar{k}[\theta(t)] < 0 \;, \tag{76}$$

implying that (26) converges to a stationary point of $\bar{k}(\cdot)$ in H.

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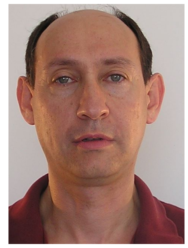

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